DESIGN OF ROBUST OBSERVER-BASED CONTROLLERS FOR LPV SYSTEMS WITH UNCERTAINTY

BY

MOHAMMED A. ALSUWAII

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This thesis, written by **MOHAMMED ABDULRAHMAN ALSUWAII** under the direction his thesis advisor and approved by his thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN ELECTRICAL ENGINEERING.**

____
Dr. Salim Ibrir
(Advisor)

____
Dr. Ali A. Al-Shaikhi
Department Chairman

____
Dr. Mahmoud Kassas
(Member)

____
Dr. Salam A. Zummo
Dean of Graduate Studies

____
Dr. Ahmed Masoud
(Member)

14/12/15
Date

Dedicated to my Father and my Mother

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Nomenclature

English Symbols

P	Lyaounov symmetric positive definite matrix
A_0	Observer system matrix
A_i	Matrices of a polytopic representation.
K	State feed-back controller gain
L	Observer gain
N	Number of vertices
q	Number of uncertain parameters
$\bigcirc_n$	Observability matrix
ω_e	Trim wind
$\frac{T_{mr}}{m}$	Trim thrust main rotor
u	Perturbation of the body axis forward speed
q	Pitch rate angle
a_1	Longitudinal flapping angle
I_{xx},I_{yy},I_{zz}	Fuselage inertias about the body-fixed frame
t	Time, sec

Greek Symbols

$\theta(t)$	Parameter vector for continuous time systems
θ_k	Parameter vector for discrete time systems
α_i	Vertex coefficients that called the unit simplex
Θ	Parameter polytope.
θ_e	Trim pitch angle
ϕ_e	Trim bank angle
X_{mr}	Main rotor force
X_{fus}	Fuselage force along the a-axis
M_{mr}	Main rotor moment
M_{ht}	Horizontal stabilizer moment along the a-axis
T_{mr}	Thrust of the main rotor
τ_e	Effective rotor time constant

Abbreviations

LMI	Linear Matrix Inequality
BMI	Bilinear Matrix Inequality
LPV	linear parameter varying

Thesis Abstract

Name:	MOHAMMED ABDULRAHMAN ALSUWAII
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The robust observer based control is very important for stabilization of uncertain systems and has became a focus of much research in recent years. New methodology of design robust observer-based control of continuous and discrete time systems for linear parameter varying(LPV) system with uncertain parameters is developed. The design based on lyapunov system theory and convex optimization approach. The parameters of the system are limited to given convex bounded so that the LPV system called polytopic. All cases of design will be formulated in terms of Linear Matrix Inequality (LMIs) which can be solved efficiently. The LMI approach is developed to construct linear full-order observer to guarantee the feedback-controlled system is exponentially stabilizable by the linear observer-based control. The new methodology developed will estimate the controller state feedback and gain observer together. During the developments, the design will be formulated as Bilinear Matrix Inequality(BMI) which are not convex have to be converted to LMIs. Two different methods are suggested to solve BMIs. Finally, the numerical examples including the LPV longitudinal model of helicopter are given to demonstrate the use *of our results. The result of development showed that observer stabilizes the system and it is convergence.*

Keywords *observer based control,linear parameter varying(LPV), Lyapunov function, Linear Matrix Inequality(LMI)*

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ملخص الرسالة

الاسم **محمد عبد الرحمن السويعي**

عنوان الرسالة **تصميم التحكم المراقب القوي للأنظمة الخطية المتغيرة المعالم**

التخصص **الهندسة الكهربائية**

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التحكم المراقب القوي مهمة جدا لاستقرار الأنظمة الغير مؤكدة (الغامضة) ولهذا أصبح محور الكثير من الأبحاث في السنوات الأخيرة. في هذه الأطروحة تم تطوير منهجيه جديدة لتصميم التحكم المراقب القوي للأنظمة الغامضة الخطية المتغيرة معلماتها للزمن المستمر والزمن المنفصل. تم التصميم بالاعتماد على نظرية ليبانوف وبالاستثمال المحدب (convex optimization) .

جميع معلمات النظام تم صياغتها على هيئة (polytopic model). ايضا جميع حالات التصميم سوف يتم تمثيلها على شكل المتراجحة الخطية للمصفوفات(LMI) والتي يمكن حلها بكفائه. تم تطوير منهجية المتراجحه الخطية للمصفوفات (LMI) لبناء مراقب يضمن استقرار نظام التغذية المرتدة بشكل مضاعف. في هذه المنهجية الجديدة يتم ايجاد كسب المتحكم و كسب المراقب مع بعض بعض .

اثناء التطوير يظهر التصميم على صيغة ثنائية الخطية المتراجحه لمصفوفات(BMI) والتي يجب تحويلها الى المتراجحة الخطية للمصفوفات(LMI) تم اقتراح طريقتين لتحويل (BMIs) الى (LMIs)

وأخيرا، وبالنظر إلى الأمثلة العددية بما في ذلك نموذج الطولي للطائرة هليكوبتر للتدليل على استخدام نتائجنا النتيجة أظهرت إن المنهجية المقترحة توصل النظام الى الاستقرار بشكل سريع .

Chapter 1
Introduction

1.1 Background

In many practical systems, the analysis of a mathematical model is usually an important work for a control engineer so as to control a system. However, the mathematical model always contains some uncertain elements. These uncertainties may be due to additive unknown internal or external noise, environmental influence, nonlinearities such as hysteresis or friction, poor plant knowledge, reduced-order models, uncertain or slowly varying parameters. The states of a system are not always measurable in many control systems . Hence the state observer will be used to reconstruct the states of a dynamic system. The robust observer-based control is probably well suited in such situations for stabilization of uncertain system. Furthermore, an efficient approach to design this robust observer based control is very important and has became a focus of much research in recent years. These motivate us to consider the observer design and robust observer-based control for uncertain systems. Recently, much effort has been devoted to design the observer or observer-based control of uncertain systems with many approaches.

Due to the variation of system properties, the performance of most systems are subject to model uncertainties. Two main classes of uncertainties called unstructured and structured uncertainties are distinguished for the system robust analysis [4]. Unstructured uncertainties, which represent the effects of nonlinearity, high frequency unmodeled dynamics and linearization truncations, are specified as a ball of norm-bounded operators in the analysis of the stability of systems. Structured uncertainties, also called parametric uncertainties, represent the lack of precise knowledge of the actual system parameters.

In this thesis, we will adopt these two useful methodologies lyapunov stability theory and LMI approach to the design of the LPV observer-based controller for a class of uncertain systems in continuous and discrete time . The control and observer gains are found from the solution of LMIs. In the recent years, the LMI have emerged as a powerful tool for approaching control problems that apear difficult if not impossible to solve. In this thesis the LMI will be solved by Yalmip toolbox . It is a very flexible toolbox for defining and solving advanced optimization problems.

1.2 Robustness Analysis And Lyapunov Function

Another important subject used to analyse robustness of uncertain systems is that of lyapunov theory, which has been extensively reported in the recent two decades. In particular, the use of one common form of lyapunov function, the quadratic lyapunov function, known as Quadratic Stability (QS), has provided wide application for the robustness analysis of uncertain systems through the use of the convex optimization method used in LMIs, which can be solved efficiently in polynomial time. The QS approach can even handle time-varying uncertain systems. However, it may lead to very conservative solutions when the uncertainties are time-invariant, since only one single function is used for the analysis of the entire uncertain system set. To reduce the conservatismone newfunction, called the parameter-dependent lyapunov function, has been extended from the QS approach as the function depends on the uncertain parameters directly. The affine parameter-dependent lyapunov function is the simplest case of this extension, which has been applied successfully to achieve several different sufficient LMI conditions for robustness analysis, including Robust Stability (RS) [2] and Extended Stability (ES) for continuous system (Hurwitz stability) and RS in and ES for discrete system tem (Schur stability) respectively. The performance of these approaches and the QS approach were compared in [6] systematically through numerical examples, and the results indicate that both the RS and the ES approaches have less conservativness than the QS, but it is difficult to compare the RS and ES, because there are cases that can be solved by the RS but not by the ES and vice-versa. Using a relaxation procedure, a sequence of LMI conditions based on affine parameter-dependent lyapunov functions were constructed [3]. As the number of LMIs increase, these sufficient conditions can become necessary as well.

A more complex function, the homogeneous parameter-dependent lyapunov function, which depends polynomially on uncertain parameters is introduced to obtain less conservative LMI conditions. Using this function presented one LMI condition whose conservatism can be reduced as the degree of the polynomials increases[4]. Using a similar structure of homogeneous function in [3] but not in [4] and in [5] proposed a less computationally intensive LMI condition. This LMI condition provides a complete characterization of robust stability as the degree of the polynomial lyapunov matrices increases. However, as the number of LMIs or the degree of polynomials increases, the complexity of theses conditions will increase dramatically and testing of them will require much more computation than those in [6] which are more conservative in certain cases. Therefore, a compromise between the complexity and the conservatism should be taken into account when applied to uncertain systems.

1.3 Literature Review

Linear Parameter Varying (LPV) systems have received considerable attention by the automatic control researchers[25]. Controllers of state feedback and observer form are one way for searching for dynamic output feedback controllers that general issue has convex LMI-based solutions as long as the systems are not affected by uncertainties. As soon as the systems are affected by uncertainties, the problem becomes more complex.

Furthermore, in the last years more researchers have paid attention to uncertainty, the robust LPV observer for discrete time uncertain system is investigated and the methodology used based on separation principle. The output-based controllers are obtained using separate design of the observer and the state feedback and it showed that the interconnection of the LPV plant, observer and state feedback leads to a stable closed-loop system for certain levels of mismatch between estimated and true parameters [36]. For continuous time, robust H-infinity estimator is investigated for polytopic uncertain systems. The purpose is to obtain a stable and proper linear full-order estimator such that estimation error system remains robustly stable with prescribed attenuation level. Based on proposed H-infinity which decouple with lyapunov matrix and existence of robust estimator is provided in terms of LMIs [52].

For systems without uncertainty, An interpolation based method has been proposed to design a parameter varying observer. It is based on the resolution of LMI conditions [16]. Also, design for interval observer for continuous time based on estimation of the state each time provide two estimation of state upper bound and lower bound [46].

In [20], the observer-based parameterized LPV L_2-gain control synthesis is considered. The resulting, observer embedded control is of no need to on-line scheduling parameter variation rate and to preliminarily approximate the bound on the parameter variation rate. Thus, the LPV synthesis proposed reduces the conservatism and increases the operation ability.

Reference [47] discussed the reduced-order robust observer using nonlinear parameter estimation for induction motors to reconstruct the non measurable state variables and mainly the rotor flux. The approach is based on the notion of robust detectability and the using of LMI to calculate the gain of the polytopic reduced-order robust observer and the identification on line of the variable parameter (the rotor time constant) using a special non linear observer.

Moreover, increasing attention has been devoted to synthesis of observer-based LPV controllers with guaranteed L_2-gain and H_2-type performance objectives. These methods are based on parameter-dependent LMI conditions that are potentially conservative if compared with the ones for a controller of unrestricted structure. Nevertheless, the online scheduling of observer-based controllers is relatively simpler and their off-line synthesis has an admissible level of extra computational complexity objectives[30].

In [45], the polytopic observers for LPV discrete-time systems is considered.

The design based on Parameter dependent lyapunov function, called poly-quadratic lyapunov function, is considered. It incorporates the parameter variations for LPV systems with polytopic parameter dependence and allows to guarantee a so-called poly-quadratic stability which is sufficient to ensure global asymptotic Stability. The methodology suggested is based on finding lyapunov function of the closed loop state feedback and observer and estimate the robust state feedback gain of control and gain of observer from applying the stability conditions on the lyapunov function. Also, the state dynamics matrix of observer is include since the system is uncertainty.

A software framework for robust optimization has been presented. The modeling language in YALMIP with solver SeDuMi. YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems. It is implemented as toolbox for MATLAB. It implements a large amount of modeling tricks, allowing the user to concentrate on the high-level model, while YALMIP takes care of the low-level modeling to obtain as efficient and numerically sound models as possible [31].

1.4 Objectives Of The Thesis

The objective of this thesis is to address the following

- Designing robust observer-based control of continuous time and discrete time for LPV systems with uncertainty. The design based on convex optimization and lyapunov system theory. The new methodology developed will estimate the controller state feed-back and gain observer together. The LPV system represented by polytpic model. .All cases of design will be expressed as linear matrix inequalities (LMIs).
- The developed methodology will be solved by two different methods to solve the BMIs of observer-based control design.

1.5 Organization Of The Thesis

This thesis is organized in a comprehensive manner so that the reader may easily follow the progress of the work. It starts by reviewing of linear parameter varying and and convex optimization to our LPV modeling of longitudinal helicopter .Then, the main part of the thesis which is the developments and results of design of observer-based control of continuous time and discrete time.

The introductory chapter, chapter one, gives an overview of the thesis work. In this chapter, uncertain systems, robustness analysis and lyapunov function are introduced. Literature review, objectives and organization of the thesis also discussed briefly.

The basic approach used in designing robust observer-based control is discussed in chapter two. The linear parameter varying systems is presented in continuous and discrete time. The convex optimization is also presented.

Chapter three presents the LPV modeling of longitudinal helicopter model. LPV model is derived from the nonlinear model of helicopter. The LPV model is with four uncertain parameters.

In chapter four, a new methodology to design of observer-based control of continuous time systems with uncertain parameters is presented. The methodology is based on lyapunov stability theory and convex optimization approach to solve problems described by LMIs. Also, two different methods is presented to solve the BMIs of observer-based control design. Numerical examples including the LPV longitudinal model of helicopter are given to illustrate the effectiveness of the proposed design results. At the end of this chapter, discussion presents the comparison between two methods.

In chapter five, the proposed methodology also is used to design of observer-based control of discrete time systems with uncertain parameters. The methodology is based on lyapunov stability theory and convex optimization approach to solve problems described by LMIs. Also, two different methods is presented to solve the BMIs of observer-based control design. Numerical examples including are given to illustrate the effectiveness of the proposed design results. At the end of this chapter, discussion resents the comparison between the proposed methodology and the exiting done for robust observer-based control. Also, the discussion include the comparison results between the two different methods.

The final chapter, chapter six, summarizes the work presented, concludes and suggests future extensions of the thesis work.

Chapter 2
Approach

2.1 Introduction

In this chapter, convex optimization and lyapunov system theory are used for the design of observer-based control of continuous time and discrete time systems. It is known that for linear time-invariant (LTI) dynamical systems can be determined by examining the system poles and eigenvalue. However, for nonlinear systems and parameter varying system, calculating eigenvalues does not decide if the system stable or unstable. In this case, one of the way in which they can find out if the system is stable or unstable is to use lyapunov theorem. Applying the lyapunov stability conditions on the systems will force the problem to be optimization problem and formulated using LMIs because the solution require to find exist a solution to matrix inequalities[45]. The convex optimization can be described as a fusion of three disciplines: optimization, convex analysis and numerical computation. It has recently become a tool of central importance in engineering, enabling the solution of very large, practical engineering problems reliably and efficiently. The convex optimization problems are also studied for which the data is not specified exactly and it is only known to belong to a given uncertainty and the constraints must hold for all possible values of the data from uncertainty. This chapter is divided as follows. Section 2.1 presents an introduction to approach used in this thesis. Section 2.2 presents the linear parameter varying systems and section 2.3 presents the convex optimization.

2.2 Linear Parameter Varying Systems

Linear Parameter Varying (LPV) systems are linear systems whose state space matrices depend on a set of time varying parameters. The LPV systems have received considerable attention by the automatic control researchers. In this section, LPV continuous and discrete time systems are considered including the notations.

2.2.1 Continuous Time LPV Models

The LPV systems are a very special class of nonlinear systems which appears to be well suited for control of dynamical systems with parameter variations. In general, LPV techniques provide a systematic design procedure for controllers and observer. This methodology allows performance, robustness and bandwidth limitations to be incorporated into a unified framework. The LPV system was introduced to distinguish between it and both LTI(linear time-invariant) and LTV (linear time varying) systems.In robust observer. The dynamic nonlinear system is represented by

$$\dot{x} = f(x, u, t) \tag{2.1}$$

where the system is a set input,output,and state variables, related by first order differential equations. The LPV plant continuous time system are described by state space equations of form

$$\begin{aligned} \dot{x} &= A(\theta(t))x + B(\theta(t))u \\ y &= C(\theta(t))x + D(\theta(t))u \end{aligned} \tag{2.2}$$

Where x,y and u denote state vector, measured output vector and control input ,respectively, and $\theta(t)$ is a vector of time varying plant parameters. Moreover,we have the following assumptions:

(1) The system state matrix $A(\theta)$ is a continuous and bounded function and depends affinely on $\theta(t)$.

(2) The immeasurable real parameters vector $\theta(t) = (\theta(t)_1, ..., \theta(t)_q)$ upper and lower bounds are known for each element such that:

$$\begin{gathered} \theta_i^{min} \leq \theta_i \leq \theta_i^{max} \\ 1 \leq i \leq q \end{gathered}$$

Where θ_i^{min} and θ_i^{max} are the lower and upper bounds of θ_i respectively. Vectors can be formed by taking each possible permutation of upper and lower bounds of elements in θ. The LPV plant is quadratically detectable and quadratically stabilizable.The matrix $A(\theta)$ can be expressed as:

$$A(\theta) = \sum_{i=1}^{N} \alpha_i(t) A_i \;\; with \; \alpha_i \geq 0 \, , \sum_{i=1}^{N} \alpha_i = 1, N = 2^q \tag{2.3}$$

Equation (4) states that vector $A(\theta)$ belongs to the convex hull formed by the vertices A_i. It also means that a vector $A(\theta)$ can be represented by a linear combination of vertices A_i such that the coefficients α_i are greater than or equal to zero and sum of the coefficients is equal to one. This construction clearly assumes that the minimum and the maximum value of each parameter A_i is known. As an example consider a linear time varying plant:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0.6 + 0.2(2 + \sin(t)) \\ -2 & -3.2 + 0.1(2 + \sin(t)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2.4}$$

If the time varying $\theta(t)$ is used to replace the time varying component $(2+\sin(t))$,the linear time varying plant can the the system will be modeled as LPV system be modelled using LPV system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0.6 + 0.2\theta(t) \\ -2 & -3.2 + 0.1\theta(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{2.5}$$

Since $\theta(t) = 2 + \sin(t)$ then $1 \leq \theta(t) \leq 3$ and N = 2 then

$$A_1(\theta(t) = 1) = \begin{bmatrix} 0 & 0.8 \\ -2 & -3.1 \end{bmatrix} \tag{2.6}$$

$$A_2(\theta(t) = 3) = \begin{bmatrix} 0 & 1.2 \\ -2 & -2.9 \end{bmatrix} \tag{2.7}$$

2.2.2 Discrete Time LPV Models

The LPV plan can be also represented by discrete time system. The systems given by

$$
\begin{aligned}
x_k &= A(\theta_k)x_k + B(\theta_k)u_k \\
y_k &= C(\theta_k)x_k + D(\theta_k)u_k
\end{aligned} \tag{2.8}
$$

with $x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^r$ the state, output and control input at discrete time $k \in \mathbb{N}$ and $\theta_k \in \mathbb{R}^L$ is a time varying parameter. The matrices $A(\theta_k) \in \mathbb{R}^{nxn}, B \in \mathbb{R}^{nxr}, C \in \mathbb{R}^{mxn}$ and $D \in \mathbb{R}^{mxr}$ have appropriate dimensions. The parameter θ_k lies in some set $\Theta \subset \mathbb{R}^L$and we assume that $A : \Theta \rightarrow \mathbb{R}^{nxn}$can be written in the polytopic form

$$
A(\theta_k) = \sum_{i=1}^{N} \xi^i(\theta_k)A_i, \tag{2.9}
$$

with $\xi := (\xi^1, ..., \xi^N)^T$ and $\sum_{i=1}^{N} \xi^i(\theta_k) = 1$ Hence, $A(\theta_k)$ lies for each $\theta_k \in \Theta$ in the convex hull [36].

2.2.3 Observability And Detectability Of LPV Systems

Observability

As far as the observability of LPV systems is concerned, the following theorem.

Theorem 2.1.1: *System (2.8) is completely observable if rank(*$\bigcirc_n(\theta_{k:k+n-1})) = n$ *for all* $k \in Z$ *.*

where $(\bigcirc_n(\theta_{k:k+n-1}))$ *is the so-called parameter varying state-observability matrix of (2.8) defined, for* $n > 1$*, as*

$$\bigcirc_n(\theta_{k:k+n-1}) = \begin{bmatrix} C \\ CA(\theta_k) \\ . \\ . \\ . \\ C\prod_{l=0}^{n-2} A(\theta_{k+n-2-l}) \end{bmatrix} \tag{2.10}$$

and $\theta_{k:k+n-1} = \theta_k, .., \theta_{k+n-1}$. For n=1,$\bigcirc_n(\theta_{k:k+n-1})$ reduces to $\bigcirc_1(\theta_{k:k}) = C$.

In other words, the concept of observability is defined similarly to the linear case when considering all possible trajectories of the parameter $\theta_k \in \Omega_k$. Actually, theorem 2.1.1 is a straightforward extension of the condition of observability stated . Hence, in theorem 2.1.1, the constraint for all $k \in Z$ can be reinterpreted in the case of LPV systems as for all $\theta_k \in \Omega_k$.

The problem lies in that the conditions are much less tractable for LPV systems than for linear systems since, in the general case, the number of trajectories of $\theta_k \in \Omega_k$, and so the number of vectors $\theta_{k:k+n-1}$ is infinite. And yet, unfortunately, the observability of the pairs (C, A_i) assigned to the vertices of the polytope DA does not necessarily induce the observability for all the pairs $(C, A(\theta_k))$. As an example, let us consider the system obeying the form (3.8) with

$$A(\theta_k) = \begin{bmatrix} 0.6 + \theta_k & 1 \\ 1 & 0 \end{bmatrix} and\, C = [1 \;\; 0.5] \tag{2.11}$$

The parameter θ_k belongs to the range [0,1]. The observability matrix is given by

$$\bigcirc_2(\theta_k, \theta_{k+1}) = \begin{bmatrix} 1 & 0.5 \\ \theta_k + 1.1 & 1 \end{bmatrix} \tag{2.12}$$

The observability matrix for the respective pairs (C, A_1) and (C, A_2), with $A_1 =$

$A(0)$ and $A_2 = A(1)$ numerically reads

$$\bigcirc_2([0, \theta_{k+1}]) = \begin{bmatrix} 1 & 0.5 \\ 1.1 & 1 \end{bmatrix} and \;\; \bigcirc_2\left([1, \theta_{k+1}]\right) = \begin{bmatrix} 1 & 0.5 \\ 2.1 & 1 \end{bmatrix} \tag{2.13}$$

The observability matrix depending exclusively on θ_k.

It is clear that rank $(\bigcirc_2([0, \theta_{k+1}]) = rank(\bigcirc_2([1, \theta_{k+1}]) = 2$. However, for $\theta_k = 0.9$, the observability matrix numerically reads

$$\bigcirc_2([0.9, \theta_{k+1}]) = \begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix} \tag{2.14}$$

and so rank$(\bigcirc_2([0.9, \theta_{k+1}])) = 1$. As a result, the two pairs (C, A_1) and (C, A_2) are observable whereas the observability is not satisfied inside the polytope DA when $\theta_k = 0.9$. A reduction of the computational cost for testing the observability rank condition of theorem 2.1.1 is most often either a hard task or merely infeasible.

Detectability

A slightly weaker notion than observability is detectability. A system is detectable if and only if all of its unobservable modes are asymptotically stable. And yet, similarly to general nonlinear systems, stability of LPV systems can match different definitions. Hence, despite the resulting conservatism, we must resort to specific ones. For instance, detectability is defined analogously to quadratic stability, that is

Theorem 2.1.2: *The LPV system (2.8) is quadratically detectable, if there exists a matrix $P = P^T > 0$ and a matrix function $L(\theta_k)$ such that:*

$$(A(\theta_k) + L(\theta_k)C)^T P + P(A(\theta_k) + L(\theta_k)C) < 0 \;\; \forall\, \theta_k \in \; \Omega_k \tag{2.15}$$

It turns out that checking for the conditions of theorem 2.1.2 is equally computationally demanding. Let us also notice that the computation of related invariant subspaces associated to the notion of detectability is not trivial. The practical use of observability and detectability is often of limited interest and these notions do not deserve in general extensive investigation.

2.3 Convex Optimization Approach

Convex optimization is a special class of mathematical optimization problems which has been studied for a century. The relevance of this mathematic branch has been recognized by its applications in many fields such as data analysis, statistics and control. It has been proven, in automatic control, that many control problems can be recast into convex optimization problem for e.g. robust control, LPV control and constrained control. An important reason for the interest of convex optimization problems relies in the practical point of view that they can be solved numerically by efficient methods (interior-point and ellipsoid methods)[7].

2.3.1 Convex Optimization Problem

A convex optimization problem is one of the form:

$$\begin{aligned} &minimize\, f_0(x) \\ &subject\ to\, f_i(x) \leq b_i, i = 1, ..., m \end{aligned} \tag{2.16}$$

where the functions $f_0, ..., f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, i.e., satisfy

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y) \tag{2.17}$$

Figure 2.1: Convex set (left) and non-convex set (right)

.

for all x, y $\in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$. The least-squares problem and linear programming problem are both special cases of the general convex optimization problem (3.10). Before introducing the convex optimization problem, let us recall some basic definitions.

Definition 2.2.1 Affine Set

The set S in the vector space X is affine if the line through any two points in S lies in S, i.e.

$$\lambda x_1 + (1-\lambda)x_2 \in S, \forall x_1, x_2 \in S \, and \, \lambda \in \mathbb{R} \tag{2.18}$$

Definition 2.2.2 Convex Set

The set S in the vector space X is convex if the line segment between any two points in S lies in S (see figure 2.1), $i.e.$

$$\lambda x_1 + (1-\lambda)x_2 \in S, \forall x_1, x_2 \in S \, and \, 0 \leq \lambda \leq 1 \tag{2.19}$$

• The intersection of any family of convex sets is convex.

• The hyperplane defined as $\{x \in \mathbb{R}^n : a^T x = b\}$ where $a \in \mathbb{R}^n$,$a \neq 0$, $b \in \mathbb{R}$ is affine and the half-space defined as $\{x \in \mathbb{R}^n : a^T x \leq b\}$ where $a \in \mathbb{R}^n$, $a \neq 0$, $b \in \mathbb{R}$ is convex.

• The intersection of finitely many hyperplanes and half-spaces results in a polyhedron. A compact polyhedron is said to be a polytope (see figure 2.2).

Definition 2.2.3 The convex hull of a set S, denoted $Co\{S\}$, is the set of all convex

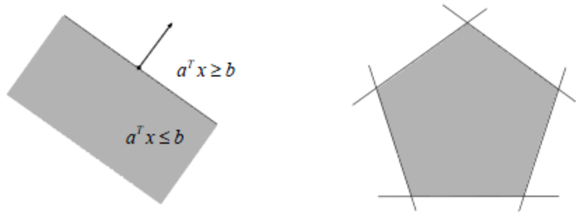

Figure 2.2: An example for the 2-dimensional space: The hyperplane (continuous line) defines 2 half-spaces (left) and a polytope defined by the intersection of 5 half-spaces (right)

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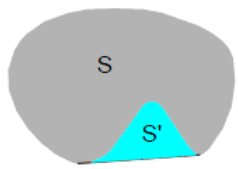

Figure 2.3: The convex hull of set S consists of all convex combinations of all elements.of S. As illustrated in this figure $Co\{S\} = S \cup S'$

combinations of points in S (see figure 2.3):

$$Co\{S\} = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n \mid x_i \in S, \lambda_i \geq 0, i = 1 : n, \sum_{i=1}^{n} \lambda_i = 1 \quad (2.20)$$

2.3.2 Solving Convex Optimization Problems

Because of their desirable properties, convex optimization problems can be solved with a variety of methods. But Interior Point or Barrier methods are especially appropriate for convex problems, because they treat linear, quadratic, conic, and smooth nonlinear functions in essentially the same way they create and use a smooth convex nonlinear barrier function for the constraints, even for LP problems. These methods make it practical to solve convex problems up to very large size, and they are especially effective on second order (quadratic and SOCP) problems, where the

Hessians of the problem functions are constant. Both theoretical results and practical experience show that Interior Point methods require a relatively small number of iterations (typically less than 50) to reach an optimal solution, independent of the number of variables and constraints (though the computational effort per iteration rises with the number of variables and constraints).

Chapter 3
LPV Helicopter Longitudinal Model

3.1 Introduction

Modeling is one of the most important parts in control engineering. It allows to mathematically represent the desired system and simulate it via appropriate simulation programs. A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on ones perspective. The dynamics of many systems, whether they are mechanical, electrical, thermal and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular systemfor example, Newtons laws for mechanical systems. This chapter is divided as follows. Section 3.1 presents an introduction to mathematical modeling of dynamic systems. Section 3.2 presents the derivation of longitudinal helicopter model. Finally, Section3.3 represents Obtaining LPV model of the longitudinal model of the helicopter.

3.2 Longitudinal Helicopter Model

The motion equation of longitudinal channel with respect to the body-fixed reference frame is deduced from newton euler equation [50]

$$\dot{u} = vr - \omega q - gsin\theta + \frac{(X_{mr} + X_{fus})}{m} \tag{3.1}$$

$$\dot{q} = pr\frac{(I_{zz} - I_{xx})}{I_{yy}} + \frac{(M_{mr} + M_{ht})}{I_{yy}} \tag{3.2}$$

The relationship between ϕ and θ is described by[50] as

$$\dot{\theta} = qcos\phi - rsin\phi \tag{3.3}$$

where U =$[u\ v\ w]^T$ and $w\ =\ [p\ q\ r]^T$ are the fuselage velocity and the angular velocity respectively in the body-fixed frame θ and ϕ are pitch angle and roll angle I_{xx}, I_{yy} and I_{zz} are fuselage inertias about the body-fixed frame X_{mr} and X_{fus} are the main rotor force and fuselage force along the x axis M_{mr} and M_{ht} are the main rotor moment and horizontal stabilizer moment along the x axis.

The main rotor is designed to be soft. It will flap when the helicopter is flying in the air. For small advance ration flight($\mu < 0.15$), one can assume that thrust vector is perpendicular to the tip-path(TTP). Thus the main rotor force and moment along x-axis can be represented as

$$X_{mr} = -T_{mr}a_1 \tag{3.4}$$

$$M_{mr} = K_{\beta} + Th_{mr}a_1 \tag{3.5}$$

where T_{mr} is the thrust of the main rotor, K_{β} is the stiffness coefficient of the restraint between the blade attachment and the rotor head h_{mr} is the vertical distance between the rotor head and helicopter center of gravity.a_1 is the longitudinal flapping angle, described as where

$$\dot{a_1} = -q - \frac{a_1}{\tau_e} + \frac{1}{\tau_e}(\frac{\partial a_1}{\partial \mu}\frac{u - u_{\omega}}{\Omega R} + \frac{\partial a_1}{\partial \mu_z}\frac{\omega - \omega_{\omega}}{\Omega R}) + \frac{A_{\delta_{lon}}}{\tau_e}\delta_{lon} \tag{3.6}$$

where τ_e is the effective rotor time constant for flapping which can be estimated as

$$\tau_e = \frac{16}{\gamma_{fb}\Omega_{mr}} \approx 0.1sec \tag{3.7}$$

$\gamma_{fb} \approx 3.7$ is Lock number. Ω = 167 rad/sec is main rotor speed, R = 0.775 is main rotor radius. In reference[50]

$$\frac{\partial a_1}{\partial \mu} = 2K_{\mu}(\frac{4\delta_{col}}{3} - \lambda_0) \tag{3.8}$$

$$\frac{\partial a_1}{\partial \mu_z} \approx K_{\mu}\frac{16\mu^2}{8|\mu| + a\sigma}sign(\mu) \tag{3.9}$$

where K_μ is scaling of flap response to speed variation, δ_{col} is the collective pitch-angle of main rotor blade, λ_0 is inflow ratio for main rotor, and the other parameter meanings can be found in reference[50]. By linearize(3.1),(3.2),(3.3) and (3.6) in the trim value of hover, we get

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \\ \dot{q} \\ \dot{a_1} \end{bmatrix} = \begin{bmatrix} X_u & -gcos\theta_e & -\omega_e & -\frac{T_{mr}}{m} \\ 0 & 0 & cos\phi_e & 0 \\ M_u & 0 & 0 & M_a \\ \frac{1}{\tau_e}\frac{1}{\Omega R}\frac{\partial a_1}{\partial \mu} & 0 & -1 & -\frac{1}{\tau_e} \end{bmatrix} \begin{bmatrix} u \\ 0 \\ -q \\ a_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 35.07 \end{bmatrix} u \tag{3.10}$$

where X_u and M_u are derivations of all forces and moments to the longitudinal velocity of the helicopter and g is acceleration of gravity.

3.3 LPV Modeling Of Longitudinal Helicopter Model

Obtaining LPV model of the longitudinal model of the helicopter following the directions in [50] is very complicated because of the complexity of the equations of motion. In this section, we show how we obtained an approximated LPV model of the helicopter. First, the parameter varying $\theta(t)$ must be specified. For the helicopter model, the aerodynamical coefficients trim pitch angle θ_e, trim bank angle ϕ_e, trim speed ω_e and trim thrust main rotor $\frac{T_{mr}}{m}$ are chosen, which are the most common choices for the longitudinal motion of helicopter. Then,the envelope of interest is chosen, that is, maximum and minimum values for $\theta(t) = [\theta_e \ \phi_e \ \omega_e \ \frac{T_{mr}}{m}]^T$. Lets consider that inside this hexadecagon convex region the helicopter can be represented by a linear combination of the 16 linear systems obtained in the corners. In other words, we can represent the behavior of the helicopter in as follows

$$\begin{aligned}\theta_{e_{min}} &\leq \theta_e \leq \theta_{e_{max}} \\ \phi_{e_{min}} &\leq \phi_e \leq \phi_{e_{max}} \\ \omega_{e_{min}} &\leq \omega_e \leq \omega_{e_{max}} \\ (\frac{T_{mr}}{m})_{min} &\leq \frac{T_{mr}}{m} \leq (\frac{T_{mr}}{m})_{max}\end{aligned} \tag{3.11}$$

and therefore

$$\begin{aligned}A(\theta) = {} & \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3 + \lambda_4 A_4 + \lambda_5 A_5 + \lambda_6 A_6 + \lambda_7 A_7 + \lambda_8 A_8 + \lambda_9 A_9 \\ & + \lambda_{10} A_{10} + \lambda_{11} A_{11} + \lambda_{12} A_{12} + \lambda_{13} A_{13} + \lambda_{14} A_{14} + \lambda_{15} A_{15} + \lambda_{16} A_{16}\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
A_1 &= A(\theta_{e_{min}}, \phi_{e_{min}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{min}), A_2 = A(\theta_{e_{min}}, \phi_{e_{min}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{max}), \\
A_3 &= A(\theta_{e_{min}}, \phi_{e_{min}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{min}), A_4 = A(\theta_{e_{min}}, \phi_{e_{min}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{max}), \\
A_5 &= A(\theta_{e_{min}}, \phi_{e_{max}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{min}), A_6 = A(\theta_{e_{min}}, \phi_{e_{max}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{max}), \\
A_7 &= A(\theta_{e_{min}}, \phi_{e_{max}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{min}), A_8 = A(\theta_{e_{min}}, \phi_{e_{max}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{max}), \\
A_9 &= A(\theta_{e_{max}}, \phi_{e_{min}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{min}), A_{10} = A(\theta_{e_{max}}, \phi_{e_{min}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{max}), \\
A_{11} &= A(\theta_{e_{max}}, \phi_{e_{min}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{min}), A_{12} = A(\theta_{e_{max}}, \phi_{e_{min}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{max}), \\
A_{13} &= A(\theta_{e_{max}}, \phi_{e_{max}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{min}), A_{14} = A(\theta_{e_{max}}, \phi_{e_{max}}, \omega_{e_{min}}, \frac{T_{mr}}{m}_{max}), \\
A_{15} &= A(\theta_{e_{max}}, \phi_{e_{max}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{min}), A_{16} = A(\theta_{e_{max}}, \phi_{e_{max}}, \omega_{e_{max}}, \frac{T_{mr}}{m}_{max})
\end{aligned} \tag{3.13}$$

It is easy to check that λ_i are convex coordinates as they satisfy (2.3)

$$\sum_{i=1}^{16} \lambda_i = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{15} + \lambda_{16} = \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_3(1-\alpha_4) + \alpha_1\alpha_2(1-\alpha_3)\alpha_4 + \alpha_1\alpha_2(1-\alpha_3)(1-\alpha_4) + \alpha_1(1-\alpha_2)\alpha_3\alpha_4 + \alpha_1(1-\alpha_2)\alpha_3(1-\alpha_4) + \alpha_1(1-\alpha_2)(1-\alpha_3)\alpha_4 + \alpha_1(1-\alpha_2)(1-\alpha_3)(1-\alpha_4) + (1-\alpha_1)\alpha_2\alpha_3\alpha_4 + (1-\alpha_1)\alpha_2\alpha_3(1-\alpha_4) + (1-\alpha_1)\alpha_2(1-\alpha_3)\alpha_4 + (1-\alpha_1)\alpha_2(1-\alpha_3)(1-\alpha_4) + (1-\alpha_1)(1-\alpha_2)\alpha_2\alpha_3\alpha_4 + (1-\alpha_1)(1-\alpha_2)\alpha_3(1-\alpha_4) + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\alpha_4 + (1-\alpha_1)(1-\alpha_2)(1-\alpha_3)(1-\alpha_4) \quad (3.14)$$

where $\alpha_1 \in [0,1]$, $\alpha_2 \in [0,1]$, $\alpha_3 \in [0,1]$ and $\alpha_4 \in [0,1]$.

The parameters X_u,M_u, $\frac{1}{\tau_e}\frac{1}{\Omega R}\frac{\partial a_1}{\partial \mu}$ is influenced badly by the longitudinal velocity, wind speed etc . It is very small and does not influenced the dynamic characteristics of the system seriously. we can get parameters range from the linearization of the nonlinear system as

$$X_u \in (-0.0321, 0.0321) \ , \ \ M_u = 0 \quad (3.15)$$

The parameters $\frac{1}{\tau_e}\frac{1}{\Omega R}\frac{\partial a_1}{\partial \mu}$ can be estimated

$$\frac{1}{\tau_e}\frac{1}{\Omega R}\frac{\partial a_1}{\partial \mu} \in (-\frac{8}{3}\frac{1}{\tau_e}\frac{1}{\Omega R}K_\mu {\delta_{col}}^T, \frac{8}{3}\frac{1}{\tau_e}\frac{1}{\Omega R}K_\mu {\delta_{col}}^T) = (-0.0076, 0.0076) \quad (3.16)$$

where $K_\mu = 0.2$, ${\delta_{col}}^T = 0.183$ rad, $\Omega = 167$ rad/s and M_u can be estimate from reference[50] in which M_u is identified and $M_u = 204$.
Comparing these values with the values obtain from linearization of nonlinear system, we can estimate parameters ranges in Table3.1 $M_a \; and \; \frac{1}{\tau_e}$

Table 3.1: Parameters Percentage

Parameter	Value	Percentage uncertainty
$1/\tau_e$	8.35	24%
M_a	212.964	24 %

Chapter 4
Design Of Observer-Based Control Of Continuous Time Systems

4.1 Introduction

The approach to design this robust observer-based control of continuous time is very important and has became a focus of much research in recent years. These motivate us to consider the observer design and robust observer-based control for uncertain systems. Recently, much effort has been devoted to design the observer-based control of continuous time of uncertain systems with many approaches. In[21], new sufficient LMI conditions guaranteeing the stability of uncertain linear systems by means of dynamic output feedbacks are presented. It is shown that the search of an observer-based controller for this class of systems is fundamentally decomposed into two main problems: robust stability with a memoryless state feedback and observer design with measured uncertainties. Under the fulfilment of the developed LMI conditions, It is shown that the observer-based problem is solvable without any need for some equality constraints or iterative computational algorithms. In[12], the robust $\mathcal{H}_\infty$ control problem of output dynamic observer-based control for a class of uncertain neutral systems is considered. The LMI optimization approach has been developed to construct the output $\mathcal{H}_\infty$ dynamic observer-based feedback control. Three classes of $\mathcal{H}_\infty$ observer-based controls are proposed. However, the conditions including the constraint of matrix equality are not in the classic LMI feasible form.In[11], observer-based controls for a class of uncertain neutral time-delay systems are considered. The asymptotic stabilization for the uncertain neutral systems is guaranteed with an observer-based feedback control. The LMI approach is used to design the observer-based feedback control system. Two classes of observer-based controls are proposed and their guaranteed costs are given. The control and observer gains are given from the LMI feasible solutions. A convex optimization problem with LMIs is formulated to design the optimal guaranteed-cost observer-based controls which minimize the guaranteed cost of the system considered. Also, in[14], a disturbance-observer-based robust controller is proposed for a class of time delay uncertain systems. To enhance the disturbance attenuation and performance robustness, the disturbance observer is designed, and it can be used to approximate the system disturbance which is generated by a linear exogenous system. Based on the output of the disturbance observer, a robust controller is presented for the time delay uncertain system, and the stability is proved of the closed-loop system using lyapunov method. In this chapter, a new methodology to design of observer-based control of continuous time and discrete time systems with uncertain parameters is proposed . The methodology is based on lyapunov stability theory and convex optimization approach to solve problems described by LMIs. With the proposed design methodology, the observer and controller gains are computed simultaneously. This chapter is divided as follows. Section 4.1 presents brief an introduction. In section 4.2,problem formulation is mention. Section 4.3, presents the design procedure of observer-based control design of continuous time systems. In section 4.4, numerical examples are given to illustrate the effectiveness of the proposed methodology results. Finally,section 4.5 presents discussion.

4.2 Problem Formulation

The aim of this chapter is to design some state observer with output $\hat{x}$ for systems given in (2.2) in order to replace the state-feedback law by $u = K\hat{x}$. The goal of this observer design is to have a closed-loop behavior as resembling as possible to the ideal state-feedback[36]. A suitable dynamic observer-based control for the system (2.2) is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu + L(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C\hat{x} \\ u(t) &= K\hat{x} \end{aligned} \tag{4.1}$$

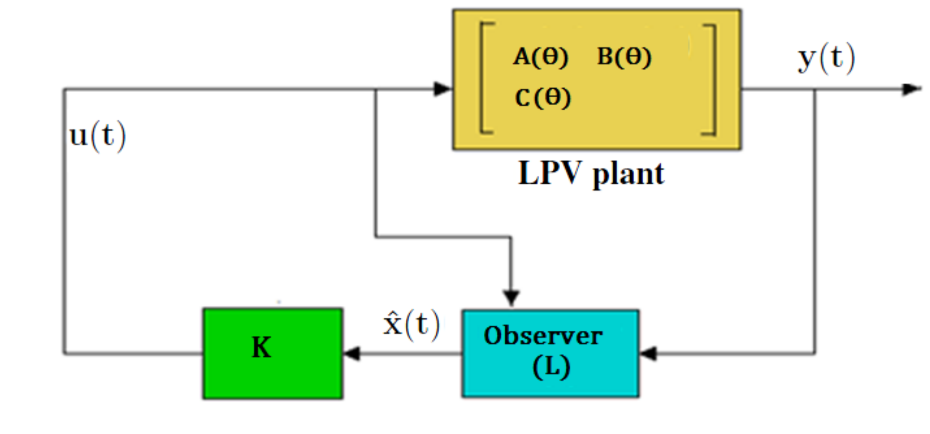

Figure 4.1: Block diagram of observer-based control for LPV systems

where the parameters to design are A , $\hat{x} \in \mathbb{R}^n$ is the estimation of x ,$\hat{y} \in \mathbb{R}^m$ is the observer output, $u \in \mathbb{R}^r$ is input vector, $K \in \mathbb{R}^{rxn}$ is the control gain and $L \in \mathbb{R}^{nxm}$ is the observer gain. The parameter A is unknown constant since the systems is uncertain. In the case of systems without uncertainties, the classical choice of $(A = A)$.

4.3 Design Procedure

In this section, we consider the design of observer-based control of continuous time systems. The procedure is as follow. First, we find the lyapunov function for closed-loop system .Then,we take derivative of lyapunov function and we apply the lyapunov stability conditions on it. Finally,the control and observer gains are formulated as the bilinear matrix inequalities (BMIs) formulations which has to be converted to LMIs since the BMI is nonconvex optimization problems. From the state observer (4.1),the estimation error $\dot{e} = \dot{x} - \dot{\hat{x}}$ is governed by

$$\dot{e}(t) = A(\theta(t))x(t) - A\hat{x} - LC(\hat{x} - x) \tag{4.2}$$

Substitute the $\hat{x} = x - e$ in (5.1) yield

$$\dot{e}(t) = (A(\theta(t)) - A)x(t) + (A - LC)e(t) \tag{4.3}$$

Now,the observer's state estimate being fed back through the state feedback gain K. Given $U = K\hat{x} = K(x - e)$ then substitute that in (2.2) yield

$$\dot{x} = (A + BK)x - BKe \tag{4.4}$$

The overall dynamics of the system combined to observation error are given by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A(\theta(t)) + BK & -BK \\ A(\theta(t)) - A & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{4.5}$$

In general $e(0) \neq 0$ but the gain force the error to approach zero. Note that if $e(t) \to 0$ as $t \to \infty$ then $\hat{x}(t) \to x(t)$ Let's assume the dynamic of observer-based control of the system (4.5) is

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{4.6}$$

where $\xi = \begin{bmatrix} x \\ e \end{bmatrix}$ Now, the lyapunov function is given by

$$V(\xi) = \xi^T P \xi \tag{4.7}$$

Then,

$$\dot{V}(\xi) = \dot{\xi}^T P \xi + \xi^T P \dot{\xi} \tag{4.8}$$

Then, we substitute the equation (4.6) into equation (4.8) as follow:

$$\dot{V}(\xi) = \begin{bmatrix} x \\ e \end{bmatrix}^T \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} x \\ e \end{bmatrix}^T P \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{4.9}$$

$$\dot{V}(\xi) = \begin{bmatrix} x \\ e \end{bmatrix}^T (\begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T P + P \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}) \begin{bmatrix} x \\ e \end{bmatrix} \tag{4.10}$$

$$\dot{V}(\xi) = \xi^T \Big(\begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T P + P \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} \Big) \xi \tag{4.11}$$

The system (4.11) is asymptotically stable if and only if there exist a symmetric positive definite matrix $P > 0$ such that

$$\Big(\begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T P \tag{4.12}$$

$$+P \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} \Big) < 0 \tag{4.13}$$

Let matrix P to be setting as $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$. Then, we substitute the matrix P in(4.13) we get

$$\begin{aligned} &\begin{bmatrix} A(\theta)^T + K^T B^T & A(\theta)^T - A_0^T \\ -K^T B^T & A_0^T - C^T L^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ &\qquad + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} < 0 \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\begin{bmatrix} A(\theta)^T P_1 + K^T B^T P_1 & A(\theta)^T P_2 - A_0^T P_2 \\ -K^T B^T P_1 & A_0^T P_2 - C^T L^T P_2 \end{bmatrix} \\ &\qquad + \begin{bmatrix} P_1 A(\theta) + P_1 BK & -P_1 BK \\ P_2 A(\theta) - P_2 A_0 & A_0 - P_2 LC \end{bmatrix} < 0 \end{aligned} \tag{4.15}$$

$$\begin{bmatrix} A(\theta)^T P_1 + K^T B^T P_1 + P_1 A(\theta) + P_1 BK & A(\theta)^T P_2 - A_0^T P_2 - P_1 BK \\ -K^T B^T P_1 + P_2 A(\theta) - P_2 A_0 & A_0^T P_2 - C^T L^T P_2 + P_2 A_0 - P_2 LC \end{bmatrix} < 0 \tag{4.16}$$

Inequality (4.16) is BMI which is nonconvex optimization problems. The problems formulated in BMI terms may have multiple local solutions and cannot be resolved using the convex optimization techniques developed for solving LMI problems, as they are often difficult nonconvex problems. Some heuristic methods which enable solutions to these types of problems have been proposed. We can solve this type of problem using linearization technique by two methods. The first method , a change in variables is used to transform the BMI yo LMI. The product of two variables is replaced by a new variable as follows:

Let $P_1 B = BZ$ and $K = Z^{-1}G$ then $P_1 BK = BZ^{-1}ZG = BG$. Similarly, $K^T B^T P_1 = G^T B^T$.Other, $X = P_2 A_0$ and $X^T = A_0^T P_2$, $Y = P_2 L$ and $Y^T = L^T P_2$ where $X \in \mathbb{R}^{nxn}$, $G \in \mathbb{R}^{rxn}$, $Y \in \mathbb{R}^{nxm}$ and Z is a scaler. The result:

$$\begin{bmatrix} A(\theta)^T P_1 + G^T B^T + P_1 A(\theta) + BG & A(\theta)^T P_2 - X^T - BG \\ -G^T B^T + P_2 A(\theta) - X & X^T - C^T Y^T + X - YC \end{bmatrix} < 0 \quad (4.17)$$

Theorem 4.3.1: *System (2.2) is exponentially stabilizable if there exist symmetric matrices $P_1 \in \mathbb{R}^{nxn}$ and $P_2 \in \mathbb{R}^{nxn}$,matrices $X \in \mathbb{R}^{nxn}$, $G \in \mathbb{R}^{rxn}$, $Y \in \mathbb{R}^{nxm}$ and Z is a scaler such that*

$$P_1 > 0 \quad (4.18a)$$

$$P_2 > 0 \quad (4.18b)$$

$$P_1 B = BZ \quad (4.18c)$$

$$\begin{bmatrix} A(\theta)^T P_1 + G^T B^T + P_1 A(\theta) + BG & A(\theta)^T P_2 - X^T - BG \\ -G^T B^T + P_2 A(\theta) - X & X^T - C^T Y^T + X - YC \end{bmatrix} < 0 \quad (4.18d)$$

Then, the stabilizing observer-based control gain is given by $K = Z^{-1}G$, observer gain $L = P_2^{-1}Y$ and $A_0 = P_2^{-1}X$.

The second method to solve (4.16) is to solve first $P1$ and K. Then, we solve for P_2 , A_0 and L to avoid getting equality relation in the first method. This method is called substitution method. According to property of LMI, the block diagonal matrices could be considered as separate constraint. The procedure as follow:

$$A(\theta)^T P_1 + K^T B^T P_1 + P_1 A(\theta) + P_1 BK < 0 \quad (4.19)$$

Let $K = Y_1 P_1$

$$A(\theta)^T P_1 + P_1 Y_1^T B^T P_1 + P_1 A(\theta) + P_1 B Y_1 P_1 < 0 \quad (4.20)$$

Multiplying both side (4.19) by P_1^{-1},we get

$$P_1^{-1}A(\theta)^T + Y_1^T B^T + A(\theta)P_1^{-1} + BY_1 < 0 \tag{4.21}$$

Let $X_1 = P_1^{-1}$. Then (4.20) rewritten as

$$X_1 A(\theta)^T + Y_1^T B^T + A(\theta)X_1 + BY_1 < 0 \tag{4.22}$$

By solving with respect to X_1 and Y_1 and putting $Y_2 = P_2 L$ and $X_2 = P_2 A_0$, we get the following matrix inequality

$$\begin{bmatrix} A(\theta)^T P_1 + K^T B^T P_1 + P_1 A(\theta) + P_1 BK \\ -K^T B^T P_1 + P_2 A(\theta) - X_2 \end{bmatrix} \begin{bmatrix} A(\theta)^T P_2 - X_2 - P_1 BK \\ X_2^T - C^T Y_2^T + X_2 - Y_2 C \end{bmatrix} < 0 \tag{4.23}$$

Finally, by solving with respect to X_2 and Y_2, we get P_1,P_2,A,K and L.

Theorem 4.3.2: *System (2.2) is exponentially stabilizable if there exist symmetric matrix $P_2 \in \mathbb{R}^{nxn}$, symmetric matrix matrix $X_1 \in \mathbb{R}^{nxn}$, matrix $X_2 \in \mathbb{R}^{nxn}$, matrix $Y_1 \in \mathbb{R}^{rxn}$ and matrix $Y_2 \in \mathbb{R}^{nxm}$ such that*

$$X_1 > 0 \tag{4.24a}$$

$$P_2 > 0 \tag{4.24b}$$

$$X_1 A(\theta)^T + Y_1^T B^T + A(\theta)X_1 + BY_1 < 0 \tag{4.24c}$$

$$\begin{bmatrix} A(\theta)^T P_1 + K^T B^T P_1 + P_1 A(\theta) + P_1 BK & A(\theta)^T P_2 - X_2^T - P_1 BK \\ -K^T B^T P_1 + P_2 A(\theta) - X_2 & X_2^T - C^T Y_2^T + X_2 - Y_2 C \end{bmatrix} < 0 \tag{4.24d}$$

Then, $P_1 = X_1^{-1}$, the control gain is given by $K = Y_1 P_1$, $A_0 = P_2^{-1} X_2$ and observer gain is given by $L = P_2^{-1} Y_2$

4.4 Numerical Examples

In the previous section, the design of observer-based control of continuous systems was presented using LMIs and lyapunov function. The design will be investigated by three examples using the simulation of the software MATLAB using the YALMIP toolbox and the solver SeDuMi. Section 4.4.1 presents example of simulation of observer-based control of continuous system with three uncertain parameters. Section 4.4.2 presents example of simulation of observer-based control for the longitudinal helicopter model. Finally, section 4.4.3 presents example of simulation of observer-based control of continuous system with all elements of system matrix are uncertain parameters.

4.4.1 First Example

Consider a continuous time LPV system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \theta_1(t) & 0 & 1 & 0 \\ 0 & \theta_2(t) & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & \theta_3(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t) \tag{4.25}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

With three uncertainties parameters $\theta_1(t)$,$\theta_2(t)$ and $\theta_3(t)$. This system is a polytopic model with i = 8 vertices. Therefore, the range of parameters as follow $\theta_1(t) \in [-5, -3]$,$\theta_2(t) \in [-3, -2]$ and $\theta_3(t) \in [-6, -5]$. This system is trivially a polytopic model with i = 8 vertices obtained by taking combinations of extremal values of the uncertainties. The following are the vertices A_i which are not stable

$$A_1 = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -5 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -6 \end{bmatrix}, A_3 = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -5 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -6 \end{bmatrix}, A_5 = \begin{bmatrix} -5 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -5 \end{bmatrix}, A_6 = \begin{bmatrix} -5 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -6 \end{bmatrix}$$

$$, \quad A_7 = \begin{bmatrix} -5 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -5 \end{bmatrix} \text{ and } A_8 = \begin{bmatrix} -5 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0.7 & 0 & 1 & 0 \\ 1 & 0 & 2 & -6 \end{bmatrix}$$

Following the procedure described in section 4.3 and theorem 4.3.1, the following matrices are obtained:

$$P_1 = \begin{bmatrix} 0.1267 & 0.0107 & -0.1025 & 0.0146 \\ 0.0107 & 0.2192 & -0.0107 & 0.0221 \\ -0.1025 & -0.0107 & 0.1267 & -0.0146 \\ 0.0146 & 0.0221 & -0.0146 & 0.0967 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.8949 & 0.0012 & -0.2314 & -0.0124 \\ 0.0012 & 0.2388 & -0.0291 & 0.0318 \\ -0.2314 & -0.0291 & 0.0913 & -0.0368 \\ -0.0124 & 0.0318 & -0.0368 & 0.1092 \end{bmatrix}$$

$A = \begin{bmatrix} 0.3747 & -0.2321 & 1.2811 & -1.2265 \\ 1.0901 & -2.3725 & 0.0102 & 0.2926 \\ 15.2229 & -0.9510 & 0.3544 & -4.8866 \\ 5.9207 & 0.3310 & 0.9498 & -6.6761 \end{bmatrix}$, robust state-feedback gain controller $K = [-15.5035 - 0.4418 - 11.4395 - 2.5462]$ and observer gains will be given by $L = \begin{bmatrix} 19.7252 \\ 6.2114 \\ 85.5332 \\ 31.1475 \end{bmatrix}$. Simulation results are shown in figures 4.2, 4.3 and 4.4. Figure 4.2 shows the real and estimated state of $x_1(t)$,$x_2(t)$,$x_3(t)$ and $x_4(t)$ and figure 4.3 displays the estimation errors of the states $x_1(t)$,$x_2(t)$,$x_3(t)$ and $x_4(t)$. They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.1. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feed-back controller stabilize the system with closed loop eigenvalue. Finally, figure 4.4 shows the input control.

Another way is to use second methodology that find the feed-back gain controller and observer gain in two stages. Following the procedure described in section 4.3 and theorem 4.3.2. First, by solving with respect to X_1 and Y_1 using LMI(4.22), the following matrices are obtained:

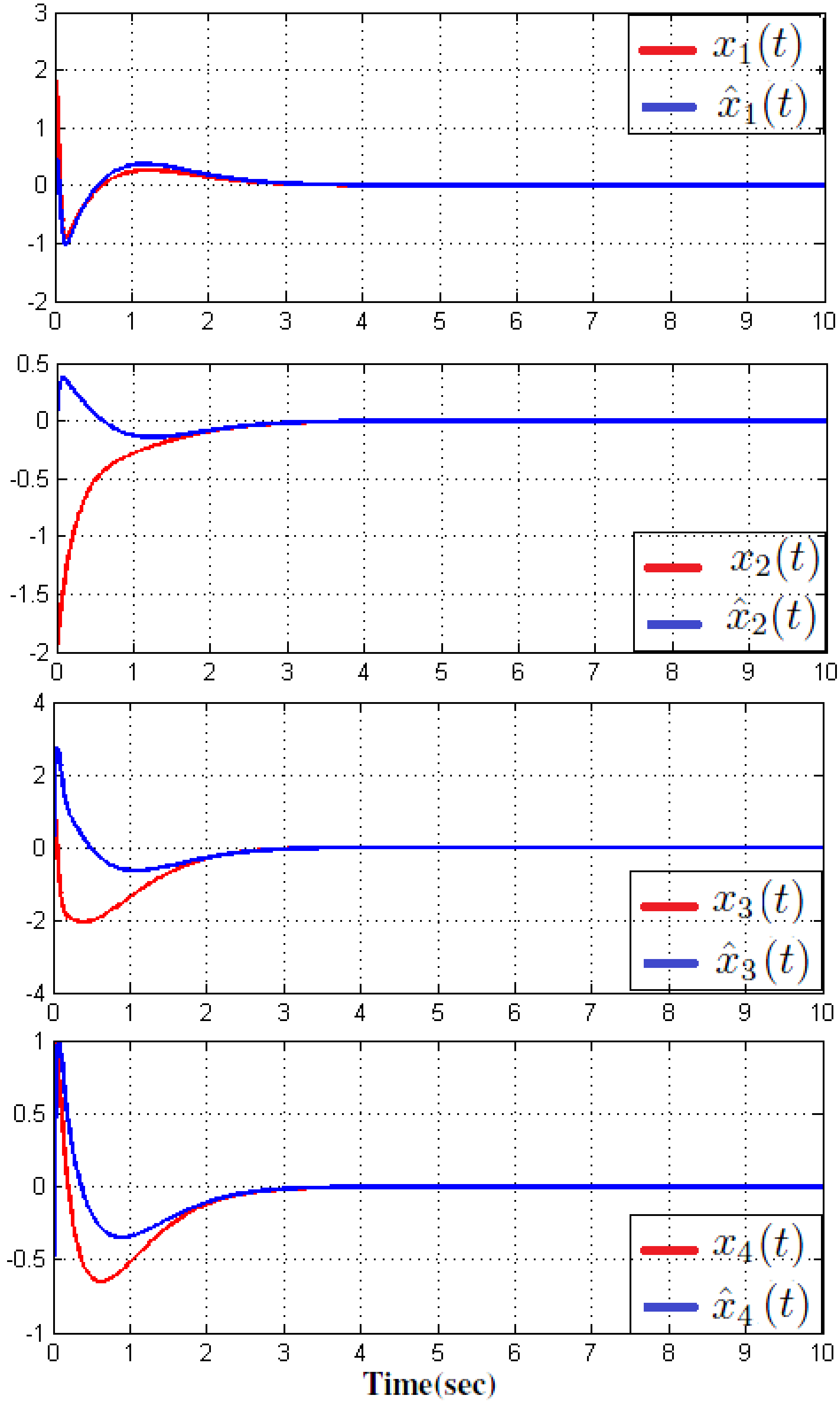

Figure 4.2: Real and estimated state response of the system

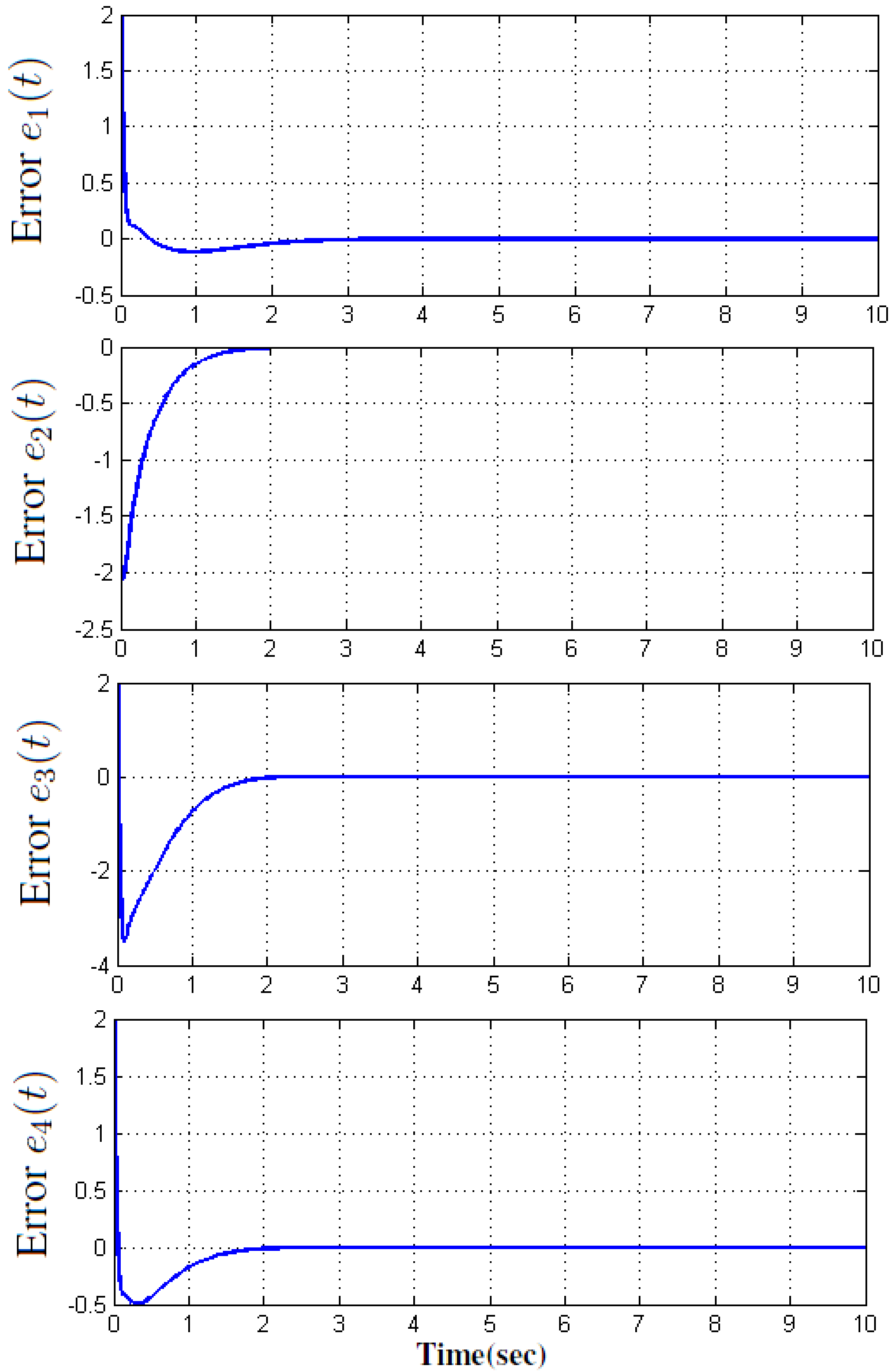

Figure 4.3: The time response error of the system

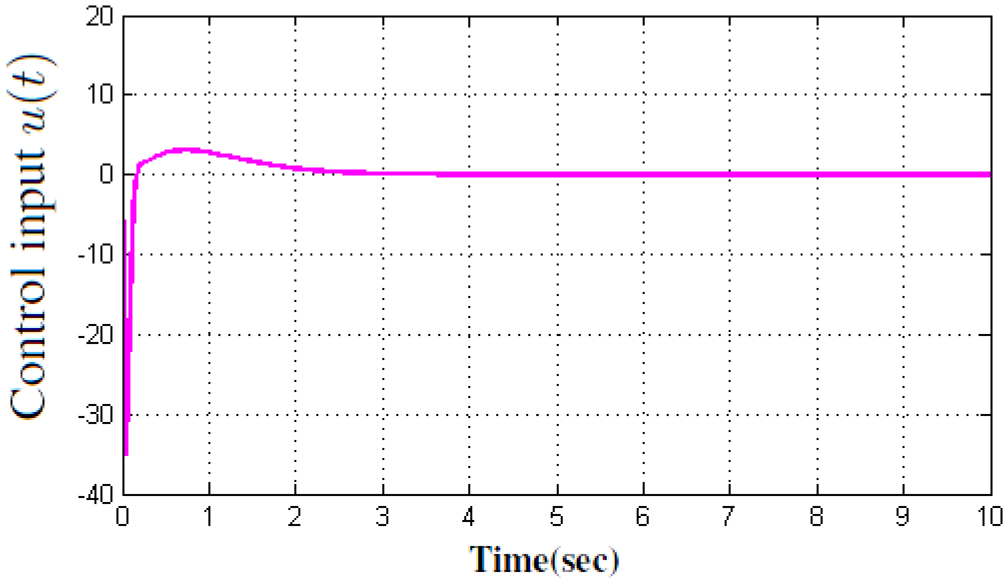

Figure 4.4: The time response of control input u(t)

Table 4.1: Eigenvalues of LMIs

Open loop system	Closed loop system
-2,-5,1.1679,-3.1679	-27.2130,-5.2341,-1.5114,-1.9845
-2,-6,1.1679,-3.1679	-27.2130,-6.2692,-1.4877,-1.9888
-3,-5,1.1679,-3.1679	-27.2131,-5.2279,-1.4933,-3.0087
-3,-6,1.1679,-3.1679	-27.1974,-6.2656,-1.4741,-3.0059
-2,-5,1.1145,-5.114	-28.5821,-5.1317,-2.1146 ±0.1246i
-2,-6,1.1145,-5.114	-28.5686,-6.166,-2.0939 ±0.1246i
-3,-5,1.1145,-5.114	-28.5821,-5.1278,-2.2593,-2.9737
-3,-6,1.1145,-5.114	-28.5821,-6.1839,-2.2068,-2.9836

P_1= $\begin{bmatrix} 6.5510 & 0.4054 & 0.0723 & 1.9419 \\ 0.4054 & 4.4734 & 0.0156 & -0.5192 \\ 0.0723 & 0.0156 & 1.7405 & -1.7848 \\ 1.9419 & -0.5192 & -1.7848 & 9.2613 \end{bmatrix}$ and K= $[-0.3733 \quad -0.0785$ $-0.8597 \quad -3.7789]$. Finally, by solving X_2 and Y_2 using LMI(4.23), we get the following matrices:

$$P_2 = \begin{bmatrix} 39.5310 & -0.5555 & -6.3815 & -4.6401 \\ -0.5555 & 13.9504 & -1.5636 & 1.3891 \\ -6.3815 & -1.5636 & 3.9937 & -3.1477 \\ -4.6401 & 1.3891 & -3.1477 & 6.9713 \end{bmatrix}$$

$$, A = \begin{bmatrix} 2.0951 & 0.1772 & 2.6780 & -1.7884 \\ 1.0907 & -2.3504 & 0.4751 & 0.0013 \\ 23.4039 & 0.5341 & 7.8372 & -8.9544 \\ 17.1347 & 0.7852 & 6.7089 & -11.0190 \end{bmatrix} \text{ and } L = \begin{bmatrix} 23.6094 \\ 5.6849 \\ 106.5661 \\ 71.8256 \end{bmatrix}$$

Table 4.2: Eigenvalues of LMIs

Open loop system	Closed loop system
-2,-5,1.1679,-3.1679	-3.3536 ±2.3424i,-1.5601, -1.9659
-2,-6,1.1679,-3.1679	-4.0030 ±2.0868i,-1.2445,-1.9826
-3,-5,1.1679,-3.1679	-3.3481 ±2.3462i,-1.5217, -3.0151
-3,-6,1.1679,-3.1679	-3.9967 ±2.0873i,-1.2264,-3.0132
-2,-5,1.1145,-5.114	-4.1981 ±1.8988i,-1.9184,- 0.2088i
-2,-6,1.1145,-5.114	-4.9502 ±2.0505i,-1.3861, -1.9465
-3,-5,1.1145,-5.114	-4.1928 ±1.8956i,-1.8809 ,-2.9664
-3,-6,1.1145,-5.114	-4.9478 ±2.0484i,-1.3518, -2.9857

Simulation results are shown in figures 4.5,4.6 and 4.7. Figure 4.5 shows the real and estimated state of $x_1(t)$,$x_2(t)$,$x_3(t)$ and $x_4(t)$ and figure 4.6 displays the estimation errors of the states $x_1(t)$,$x_2(t)$,$x_3(t)$ and $x_4(t)$. They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.2. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feed-back controller stabilize the system with closed loop eigenvalue. Finally, figure 4.7 shows the input control.

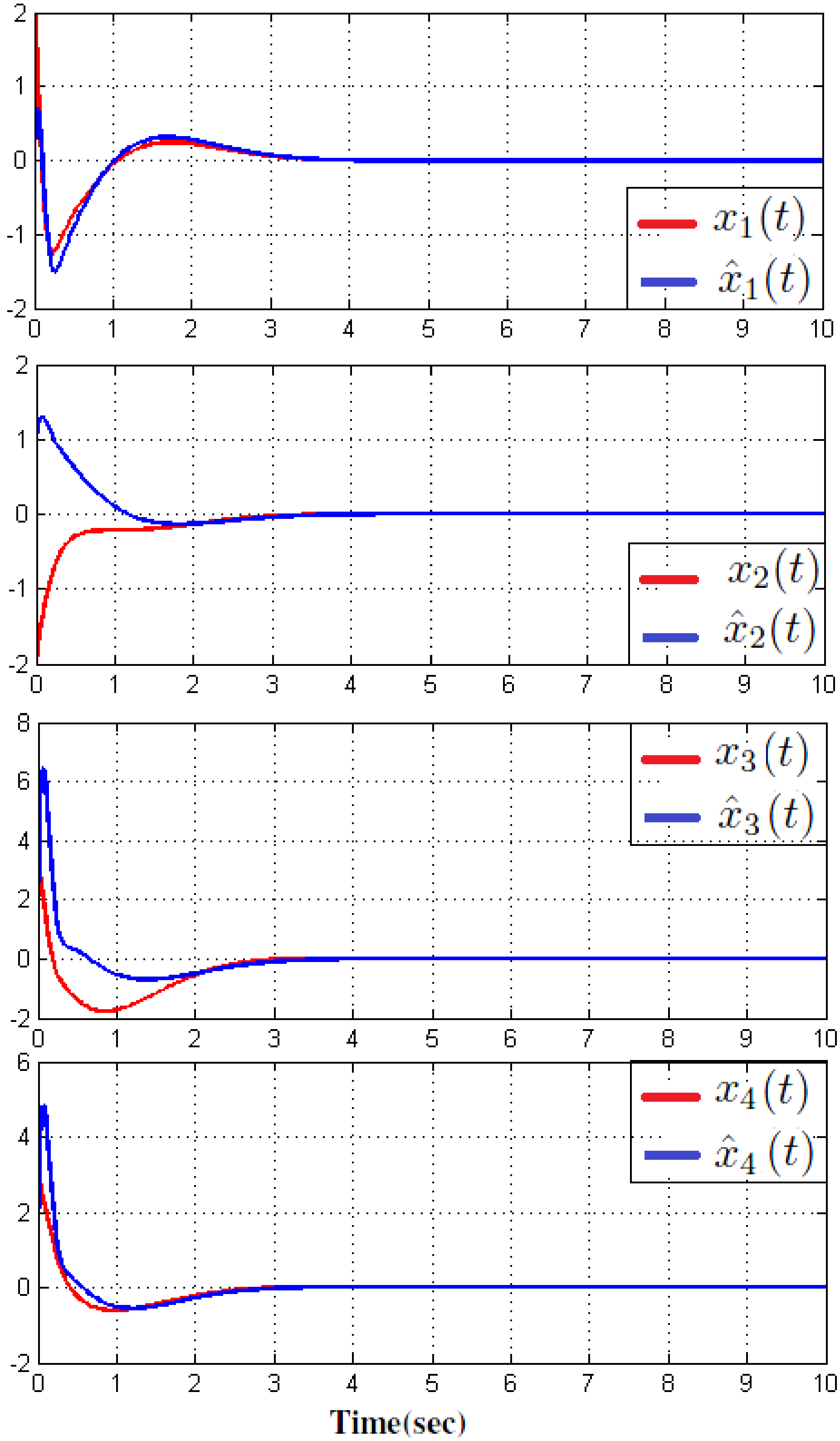

Figure 4.5: Real and estimated state response of the system

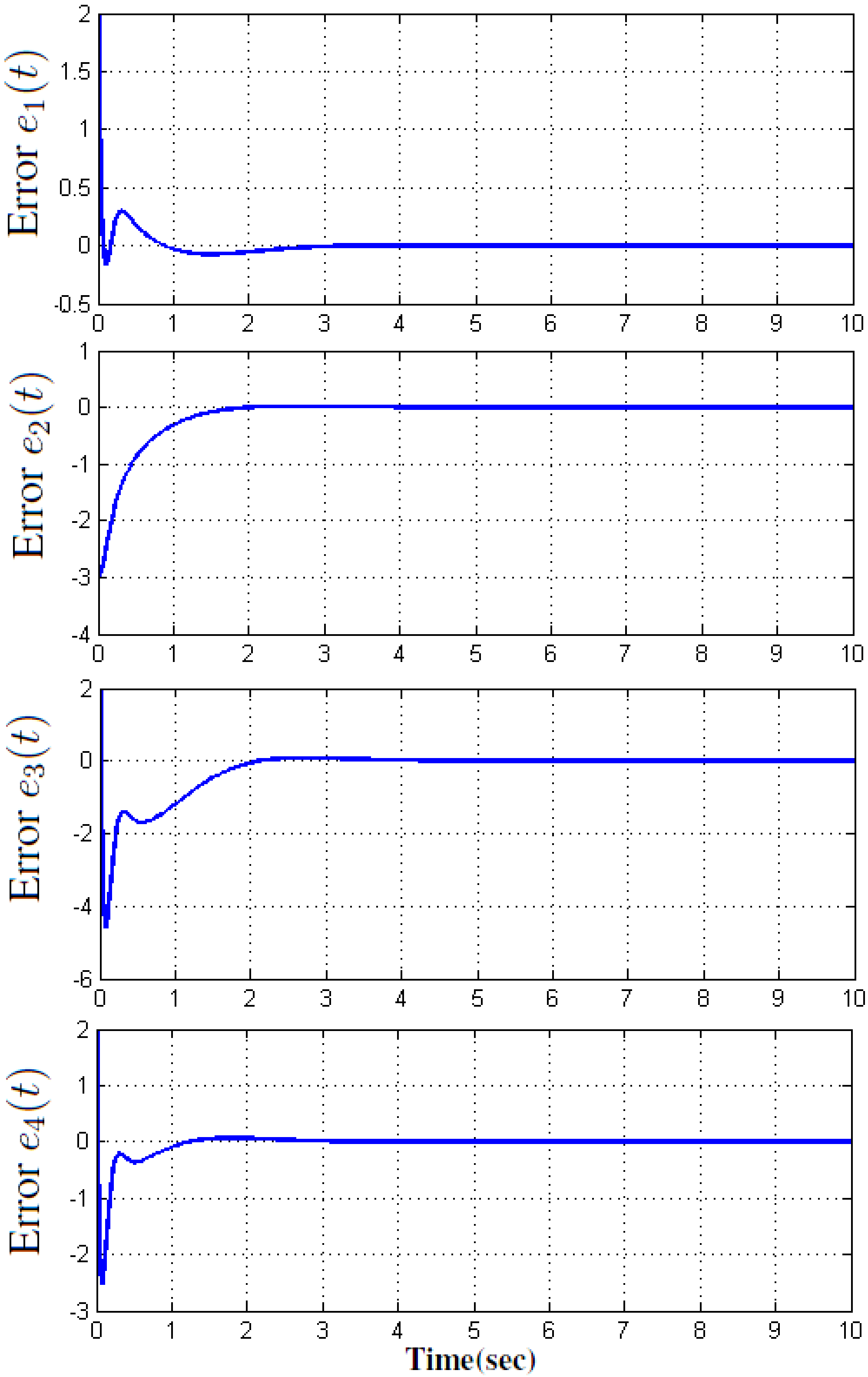

Figure 4.6: The time response error of the system

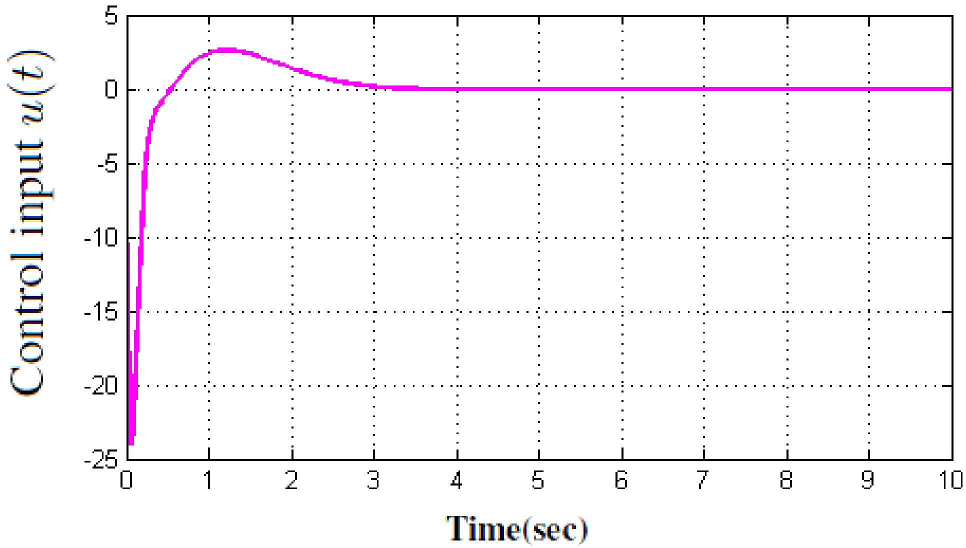

Figure 4.7: The time response of control input u(t)

4.4.2 Second Example

Consider the following helicopter longitude model:

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \\ \dot{q} \\ \dot{a}_1 \end{bmatrix} = \begin{bmatrix} 0.0321 & -gcos\theta_e(t) & -\omega_e(t) & -\frac{T_m r}{m}(t) \\ 0 & 0 & cos\phi_e(t) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ a_1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0.7 & 4 \end{bmatrix} \begin{bmatrix} \omega \\ u \end{bmatrix} \tag{4.26}$$

$$\begin{bmatrix} u \\ \theta \\ q \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ a_1 \end{bmatrix} \tag{4.27}$$

As mention in the chapter 3, the model is with four uncertain parameters trim pitch angle $\theta_e(t)$, trim bank angle $\phi_e(t)$, trim wind $\omega_e(t)$ and trim thrust main rotor $\frac{T_{mr}}{m}(t)$. The range of parameters as follow: $\theta_e \in [-15°, -15°]$, $\phi_e(t) \in [-15°, -15°]$, $\omega(t) \in [-5, 5]$ and $\frac{T_{mr}}{m}(t) \in [7.84, 11.76]$[50]. This system is trivially a polytopic model with i = 16 vertices obtained by taking combinations of extremal values of the uncertainties. The following are the vertices.

$$A_1 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & 5 & 7.84 \\ 0 & 0 & gcos(-15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & 5 & 11.76 \\ 0 & 0 & gcos(-15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & -5 & 7.84 \\ 0 & 0 & gcos(-15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & -5 & 11.76 \\ 0 & 0 & gcos(-15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & 5 & 7.84 \\ 0 & 0 & gcos(15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & 5 & 11.76 \\ 0 & 0 & gcos(15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0.0321 & -gcos(-15^{\circ}) & -5 & 7.84 \\ 0 & 0 & gcos(15^{\circ}) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 0.0321 & -gcos(-15°) & -5 & 11.76 \\ 0 & 0 & gcos(15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 0.0321 & -gcos(-15°) & 5 & 7.84 \\ 0 & 0 & gcos(-15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0.0321 & -gcos(15°) & 5 & 11.76 \\ 0 & 0 & gcos(-15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0.0321 & -gcos(15°) & -5 & 7.84 \\ 0 & 0 & gcos(-15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0.0321 & -gcos(15°) & -5 & 11.76 \\ 0 & 0 & gcos(-15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0.0321 & -gcos(15°) & 5 & 7.84 \\ 0 & 0 & gcos(15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{14} = \begin{bmatrix} 0.0321 & -gcos(15°) & 5 & 11.76 \\ 0 & 0 & gcos(15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix},$$

$$A_{15} = \begin{bmatrix} 0.0321 & -gcos(15°) & -5 & 7.84 \\ 0 & 0 & gcos(15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix}$$

and $A_{16} = \begin{bmatrix} 0.0321 & -gcos(15°) & -5 & 11.76 \\ 0 & 0 & gcos(15°) & 0 \\ 0 & 0 & 0 & 204 \\ 0.076 & 0 & -1 & -8.36 \end{bmatrix}$,

Following the procedure described in section 4.3 and theorem 4.3.1 , the following matrices are obtained:

$$P_1 = \begin{bmatrix} 0.0568 & 0.0453 & -0.0105 & -0.0061 \\ 0.0453 & 0.3003 & 0.0056 & -0.0648 \\ -0.0105 & 0.0056 & 0.0197 & 0.0150 \\ -0.0061 & -0.0648 & 0.0150 & 0.0612 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0495 & -0.0031 & -0.0133 & 0.0072 \\ -0.0031 & 0.7788 & 0.0230 & 0.4126 \\ -0.0133 & 0.0230 & 0.0059 & 0.0206 \\ 0.0072 & 0.4126 & 0.0206 & 0.7310 \end{bmatrix}$$

$A = \begin{bmatrix} 53.6416 & 44.4990 & 96.9793 & 209.0242 \\ 0.0017 & -4.6926 & -7.9427 & -16.9178 \\ 142.2638 & 198.5501 & 364.8368 & 994.8478 \\ -8.5806 & -1.6612 & -3.5515 & -18.9316 \end{bmatrix}$, robust state-feedback gain controller $K = \begin{bmatrix} -11.2800 & 8.2916 & -0.0770 & 57.0110 \\ 0.3338 & -1.3575 & -5.8177 & -25.1213 \end{bmatrix}$ and observer gains will be given by $L = \begin{bmatrix} 92.5213 & 42.6141 & 193.4428 & 207.3371 \\ -1.5809 & -3.5814 & -14.1282 & -17.3676 \\ 241.8859 & 191.3402 & 722.4282 & 988.5118 \\ -10.8666 & -2.0662 & -11.0991 & -17.7415 \end{bmatrix}$.

Simulation results are shown in figures 4.8, 4.9 and 4.10. Figure 4.8 shows the

Table 4.3: Eigenvalues of LMIs

Open loop system	Closed loop system
-4.1577±13.6510i,-0.2703, 0.2578	-39.3685 ±56.3946i,-12.7342 , -1.7070
-4.1577±13.6510i,-0.2703,0.2578	-39.3685 ±56.3946i,-12.7342 , -1.7070
-4.1994 ±13.6639i, 0.3014, -0.2305	-40.6894 ±57.1282i,-10.1322, -1.6671
-4.1994 ±13.6639i, 0.3014, -0.2305	–40.6894 ±57.1282i,-10.1322, -1.6671
-4.1577 ±13.6500i,-0.2704 ,0.2578	-39.3728 ±56.1667i,-12.7148 , -1.7177
-4.1577 ±13.6500i,-0.2704, 0.2578	-39.3728 ±56.1667i,-12.7148 , -1.7177
-4.1994 ±13.6628i,0.3014 ,-0.2305	-40.7016 ±56.9077i,-10.0948,-1.6802
-4.1994 ±13.6628i,0.3014 , -0.2305	-40.7016 ±56.9077i,-10.0948,-1.6802
-4.1577±13.6510i,-0.2703 , 0.2578	-39.3685 ±56.3946i,-12.7342 , -1.7070
-4.1577 ±13.6510i,-0.2703, 0.2578	-39.3685 ±56.3946i,-12.7342 , -1.7070
-4.1994 ±13.6639i, 0.3014,-0.2305	-40.6894 ±57.1282i,-10.1322, -1.6671
-4.1994 ±13.6639i, 0.3014, -0.2305	-40.6894 ±57.1282i,-10.1322, -1.6671
-4.1577 ±13.6500i,-0.2704, 0.2578	-39.3728 ±56.1667i,-12.7148 , -1.7177
-4.1577 ±13.6500i,-0.2704,0.2578	-39.3728 ±56.1667i,-12.7148 , -1.7177
-4.1994 ±13.6628i, 0.3014, -0.2305	-40.7016 ±56.9077i,-10.0948,-1.6802
-4.1994 ±13.6628i, 0.3014, -0.2305	-40.7016 ±56.9077i,-10.0948,-1.6802

real and estimated state of perturbation of the body axis forward speed $u(t)$, pitch angle $\theta(t)$, pitch rate $q(t)$ and figure 4.9 displays the estimation errors of the states. They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.3. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feed-back controller stabilize the system with closed loop eigenvalue. Finally, Figure 4.10 shows the fuselage velocity u and angular velocity w input control. Another way is to use second methodology that find the feed-back gain controller and observer gain in two stages.Following the procedure described in section 4.3 and theorem 4.3.2. First, by solving with respect to X_1 and Y_1 using LMI(4.22), the following matrices are obtained:

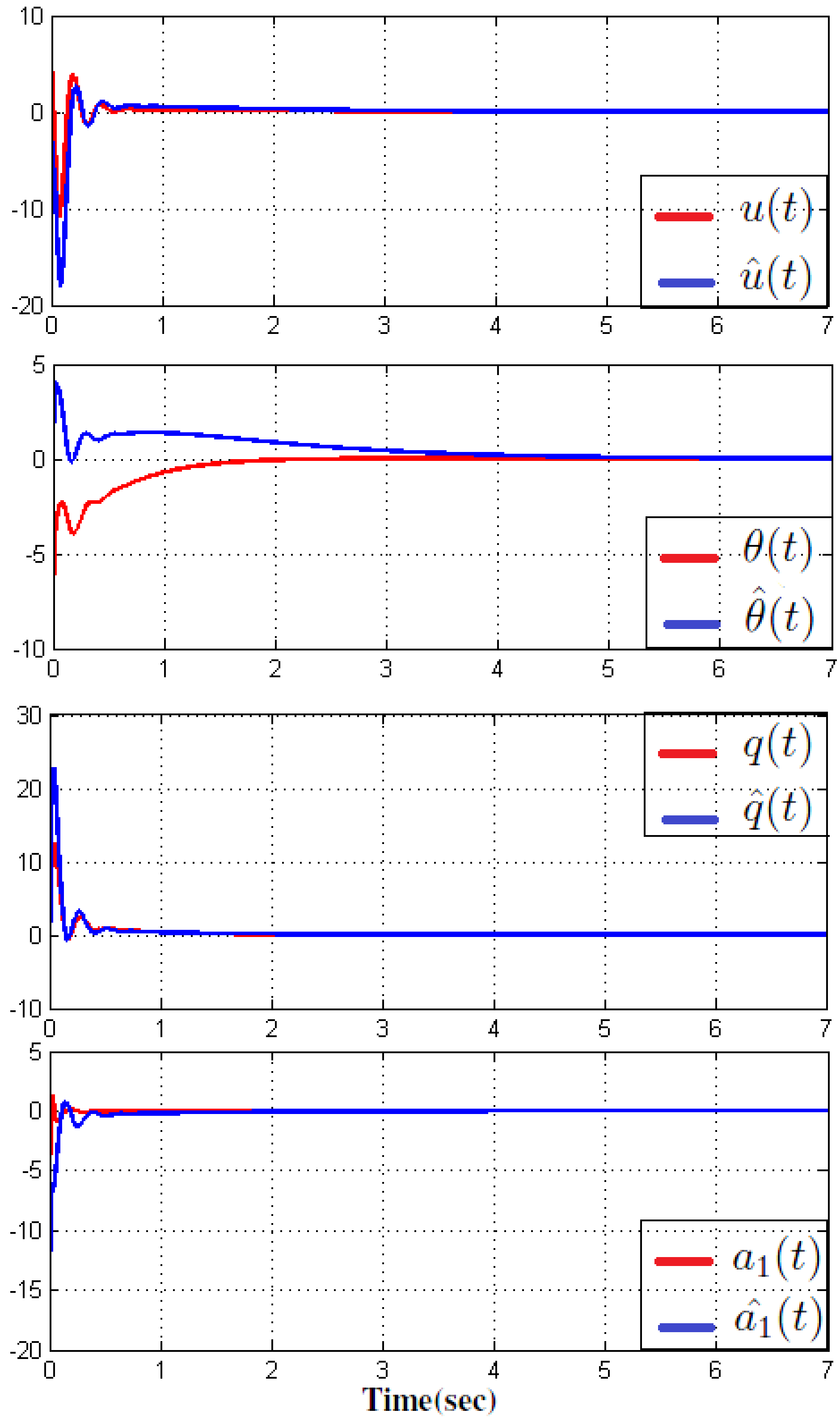

Figure 4.8: Real and estimated state response of the system

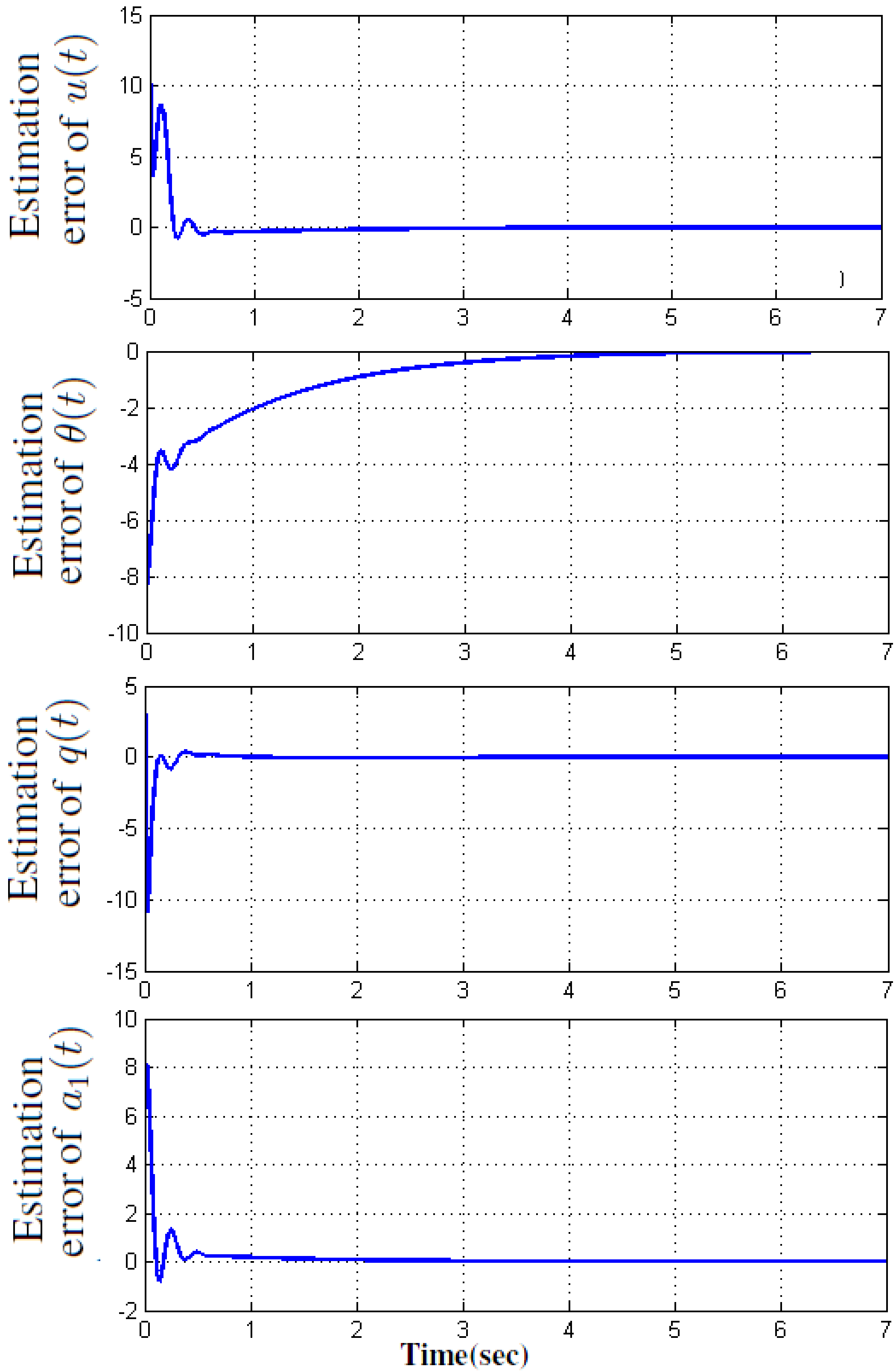

Figure 4.9: The time response error of the system

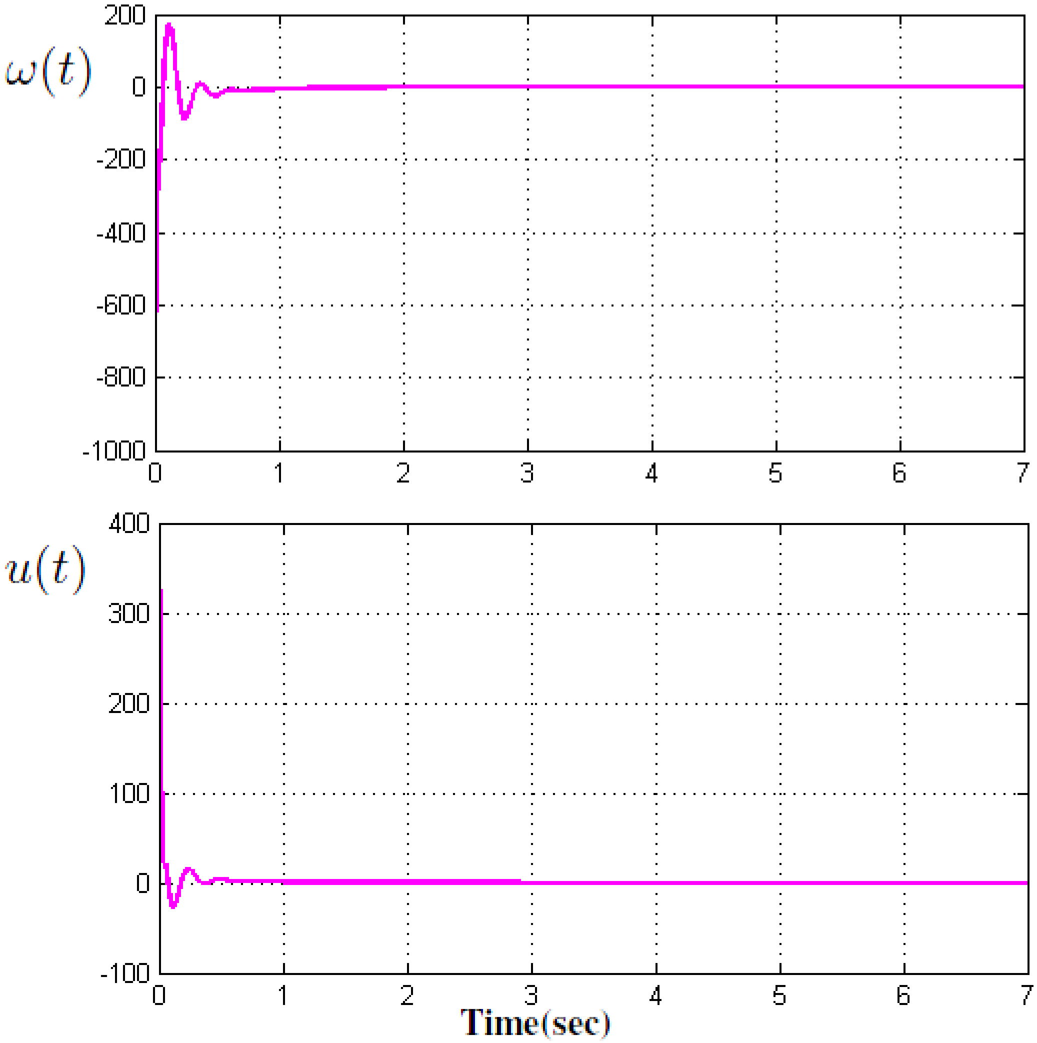

Figure 4.10: The time response of control input $w(t)$ and $u(t)$

$P_1 = \begin{bmatrix} 0.8103 & 0.3140 & 0.0885 & -0.5802 \\ 0.3140 & 2.4440 & 0.0088 & -0.8495 \\ 0.0885 & 0.0088 & 39.8452 & -6.4041 \\ -0.5802 & -0.8495 & -6.4041 & 27.8294 \end{bmatrix}$ and robust state-feedback gain controller $K = \begin{bmatrix} -0.7539 & 8.4451 & -227.0688 & 24.2628 \\ 0.2178 & -1.3020 & -36.6964 & 6.0705 \end{bmatrix}$. Finally, by solving X_2 and Y_2 using LMI(4.23), we get the following matrices:

$$P_2 = \begin{bmatrix} 43 & 302 & 11 & 195 \\ 302 & 41681 & 1350 & 33125 \\ 11 & 1350 & 104 & 1093 \\ 195 & 33125 & 1093 & 28219 \end{bmatrix}$$

, $A = \begin{bmatrix} 0.6919 & -6.6230 & 9.3425 & -27.4346 \\ -0.1118 & -0.1475 & -7.8024 & 1.7580 \\ 2.1099 & 1.1526 & 49.1879 & 258.7304 \\ 0.0394 & 0.1106 & 7.7267 & -12.5615 \end{bmatrix}$ and observer gains

$$L = \begin{bmatrix} 1921.4 & -100.9 & -90.8 & 53.0 \\ -48.7 & 33.3 & -28.2 & -42.4 \\ -101 & -60.8 & 971.7 & 241.7 \\ 47.8 & -36.5 & -3.3 & 42.6 \end{bmatrix}$$

Simulation results are shown in figures 4.11,4.12,4.13,4.14 and 4.15. Figure 4.11 shows the real and estimated state of perturbation of the body axis forward speed $u(t)$ and pitch angle $\theta(t)$ and figure 4.12 display the real and estimated state of the pitch rate $q(t)$. Also, figure 4.13 shows the real and estimated state of longitudinal flapping angle $a_1(t)$. Figure 4.14 display the estimation errors of $u(t)$, $\theta(t)$, $q(t)$ and $a_1(t)$ respectively. They shows that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.4. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feedback controller stabilize the system with closed loop eigenvalue. Finally, Figure 4.15 show and angular velocity w and the

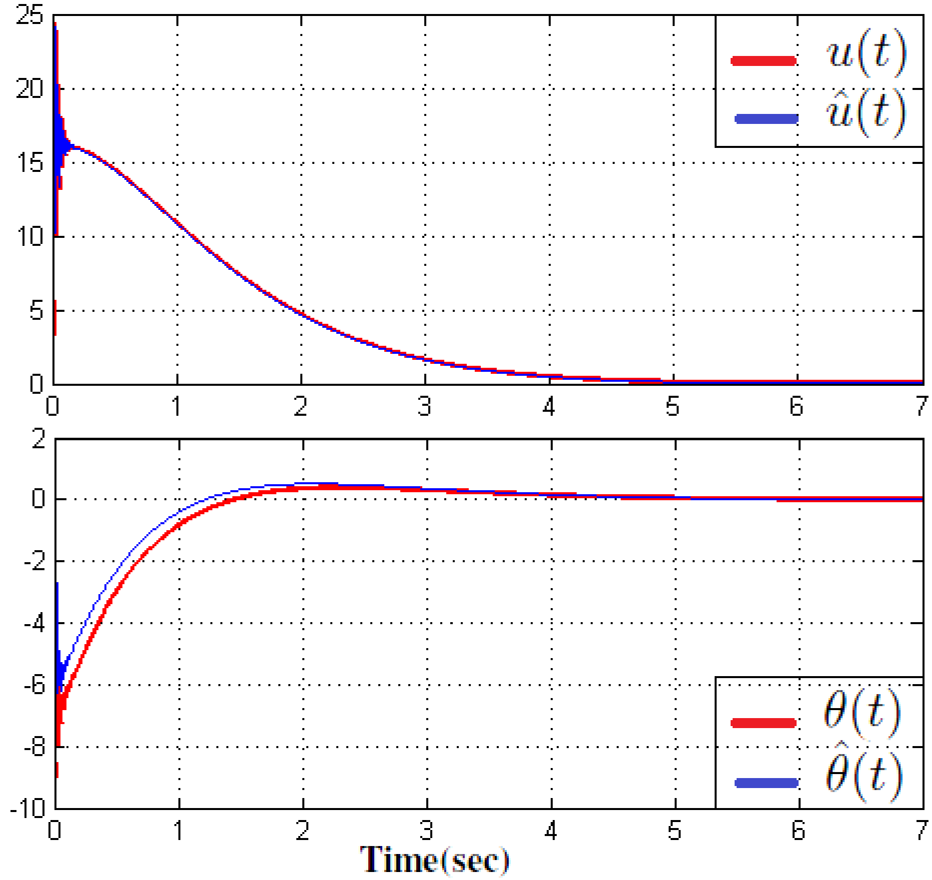

Figure 4.11: Real and estimated state of perturbation of the body axis forward speed $u(t)$ and pitch angle $\theta(t)$

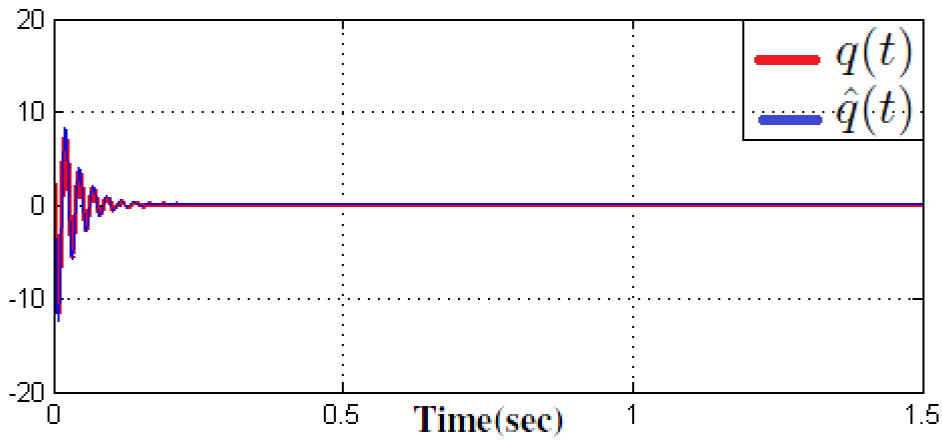

Figure 4.12: Real and estimated state of the pitch rate $q(t)$

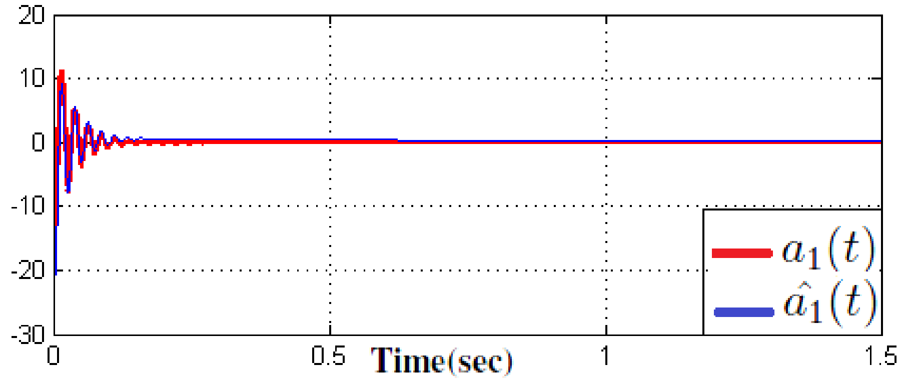

Figure 4.13: Real and estimated state response of the flapping angle $a_1(t)$

fuselage velocity u input control.

Table 4.4: Eigenvalues of LMIs

Open loop system	Closed loop system
-4.1577±13.6510i,-0.2703, 0.2578	-20.10 ±251.91i,-1.15 ±0.50i
-4.1577±13.6510i,-0.2703,0.2578	-20.10 ±251.91i,-1.15 ±0.50i
-4.1994 ±13.6639i, 0.3014, -0.2305	-20.10 ±251.92i,-1.16 ±0.51i
-4.1994 ±13.6639i, 0.3014, -0.2305	-20.10 ±251.92i,-1.16 ±0.51i
-4.1577 ±13.6500i,-0.2704 ,0.2578	-20.11 ±251.92i,-1.15 ±0.51i
-4.1577 ±13.6500i,-0.2704, 0.2578	-20.11 ±251.92i,-1.15 ±0.51i
-4.1994 ±13.6628i,0.3014 ,-0.2305	-20.10 ±251.93i,-1.15 ±0.51i
-4.1994 ±13.6628i,0.3014 , -0.2305	-20.10 ±251.93i,-1.15 ±0.51i
-4.1577±13.6510i,-0.2703 , 0.2578	-20.10 ±251.91i,-1.15 ±0.50i
-4.1577 ±13.6510i,-0.2703, 0.2578	-20.10 ±251.91i,-1.15 ±0.50i
-4.1994 ±13.6639i, 0.3014,-0.2305	-20.10 ±251.92i,-1.16 ±0.51i
-4.1994 ±13.6639i, 0.3014, -0.2305	-20.10 ±251.92i,-1.16 ±0.51i
-4.1577 ±13.6500i,-0.2704, 0.2578	-20.11 ±251.92i,-1.15 ±0.51i
-4.1577 ±13.6500i,-0.2704,0.2578	-20.11 ±251.92i,-1.15 ±0.51i
-4.1994 ±13.6628i, 0.3014, -0.2305	-20.11 ±251.93i,-1.15 ±0.51i
-4.1994 ±13.6628i, 0.3014, -0.2305	-20.11 ±251.93i,-1.15 ±0.51i

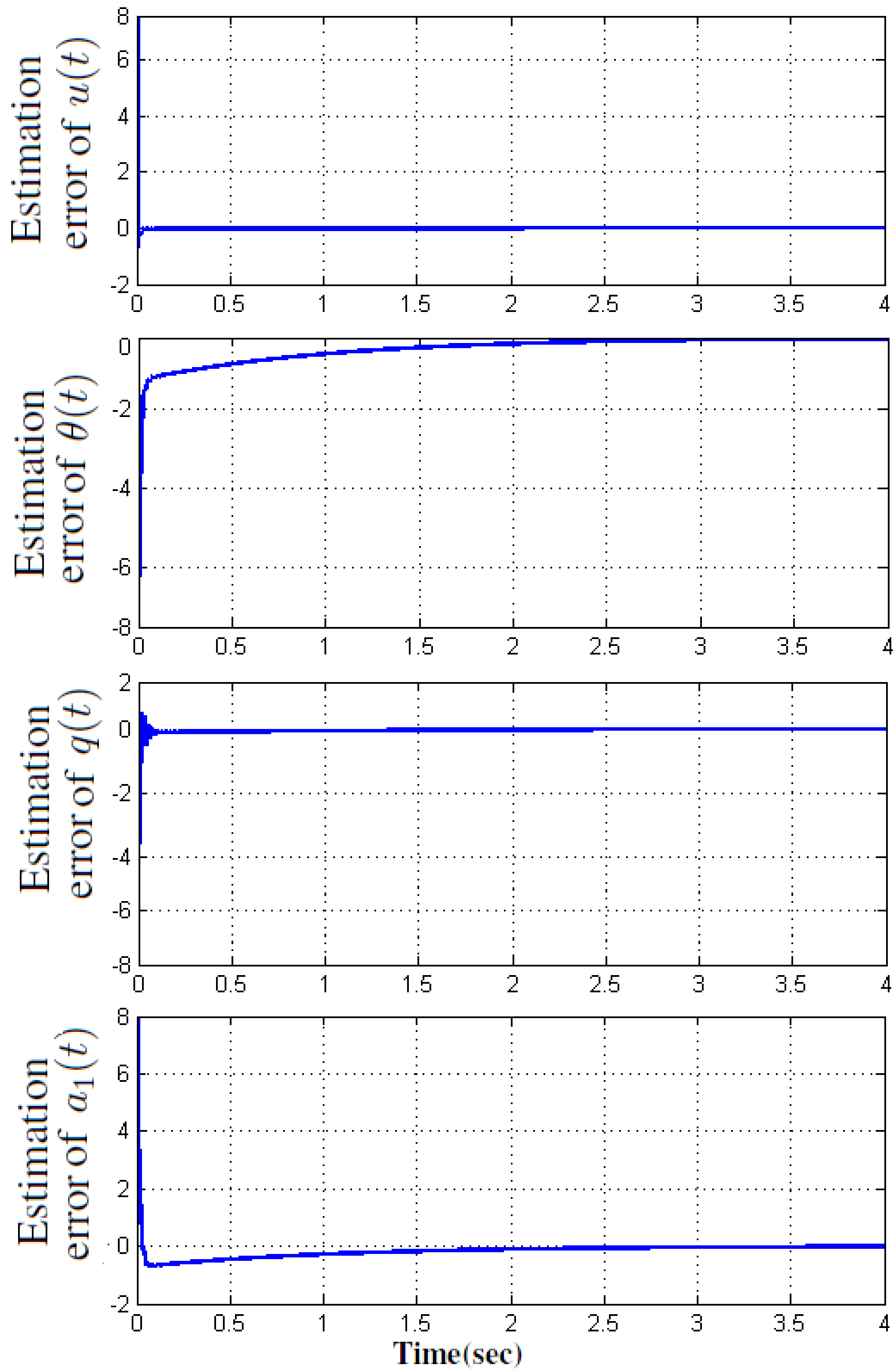

Figure 4.14: The time response error of the system

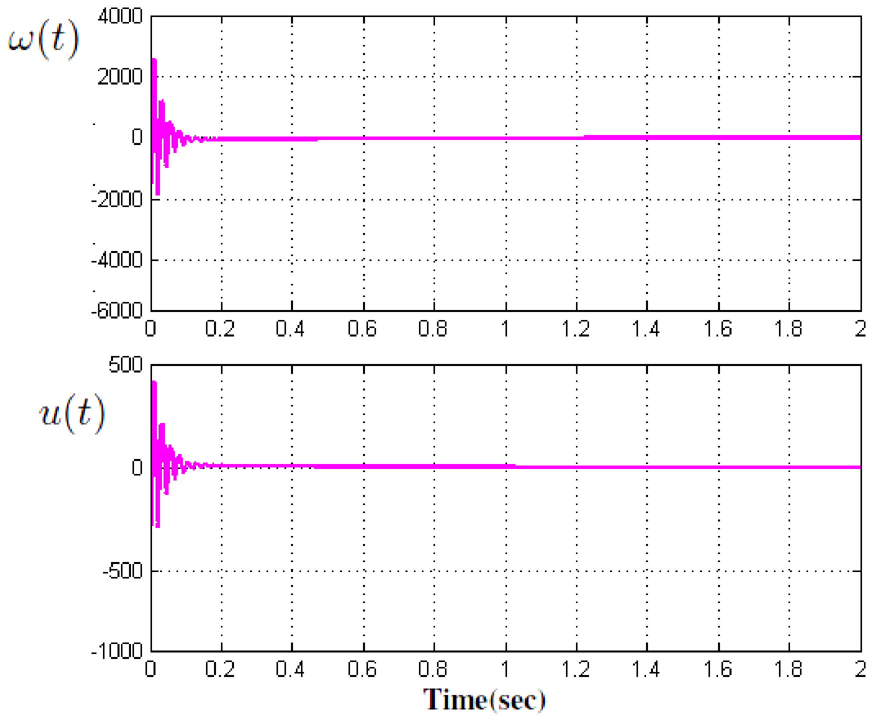

Figure 4.15: The time response of control input $w(t)$ and $u(t)$

4.4.3 Third Example

Consider a continuous time LPV system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} \theta_1(t) & \theta_2(t) \\ \theta_3(t) & \theta_4(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4.28}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with four uncertain parameters θ_1 ,θ_2,θ_3 and θ_4. As shown, all system matrix elements are uncertain parameters. Therefore, the range of parameters as follow $\theta_1 \in [-6, -3]$,$\theta_2 \in [1, 2.4]$, $\theta_3 \in [10, 20]$ and $\theta_4 \in [0, 0.7]$. This system is trivially a polytopic model with i = 16 vertices obtained by taking combinations of extremal values of the uncertainties. The following are the vertices A_i which are not stable

$$A_1 = \begin{bmatrix} -6 & 1 \\ 10 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -6 & 1 \\ 10 & 0.7 \end{bmatrix}, A_3 = \begin{bmatrix} -6 & 1 \\ 20 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} -6 & 1 \\ 20 & 0.7 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -6 & 2.4 \\ 10 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} -6 & 2.4 \\ 10 & 0.7 \end{bmatrix}, A_7 = \begin{bmatrix} -6 & 2.4 \\ 20 & 0 \end{bmatrix} \text{ and } A_8 = \begin{bmatrix} -6 & 2.4 \\ 20 & 0.7 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} -3 & 1 \\ 10 & 0 \end{bmatrix}, A_{10} = \begin{bmatrix} -3 & 1 \\ 10 & 0.7 \end{bmatrix}, A_{11} = \begin{bmatrix} -3 & 1 \\ 20 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} -6 & 1 \\ 20 & 0.7 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} -3 & 2.4 \\ 10 & 0 \end{bmatrix}, A_{14} = \begin{bmatrix} -3 & 2.4 \\ 10 & 0.7 \end{bmatrix}, A_{15} = \begin{bmatrix} -3 & 2.4 \\ 20 & 0 \end{bmatrix} \text{ and } A_{16} = \begin{bmatrix} -3 & 2.4 \\ 20 & 0.7 \end{bmatrix}$$

Following the procedure described in section 4.3 and theorem 4.3.1, the following matrices are obtained:

$$P_1 = \begin{bmatrix} 0.1230 & 0 \\ 0 & 0.0189 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0961 & -0.0036 \\ -0.0036 & 0.0369 \end{bmatrix} A = \begin{bmatrix} -3.6979 & 6.8097 \\ 15.8543 & 15.8519 \end{bmatrix},$$

robust state-feedback gain controller $K = [-24.8804 \quad -27.7898]$ and observer gains will be given by $L = \begin{bmatrix} 2.2091 \\ 29.6222 \end{bmatrix}$. Simulation results are shown in figures 4.16,4.17,4.18 and 4.19. Figure 4.16 shows the real and estimated state of $x_1(t)$ and $x_2(t)$. Figure 4.17 and figure 4.18 displays the estimation errors of the states $x_1(t)$ and $x_2(t)$ respectively. They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.5. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feedback controller stabilize the system with closed loop eigenvalue. Finally, Figure 4.19 shows the input control. Another way

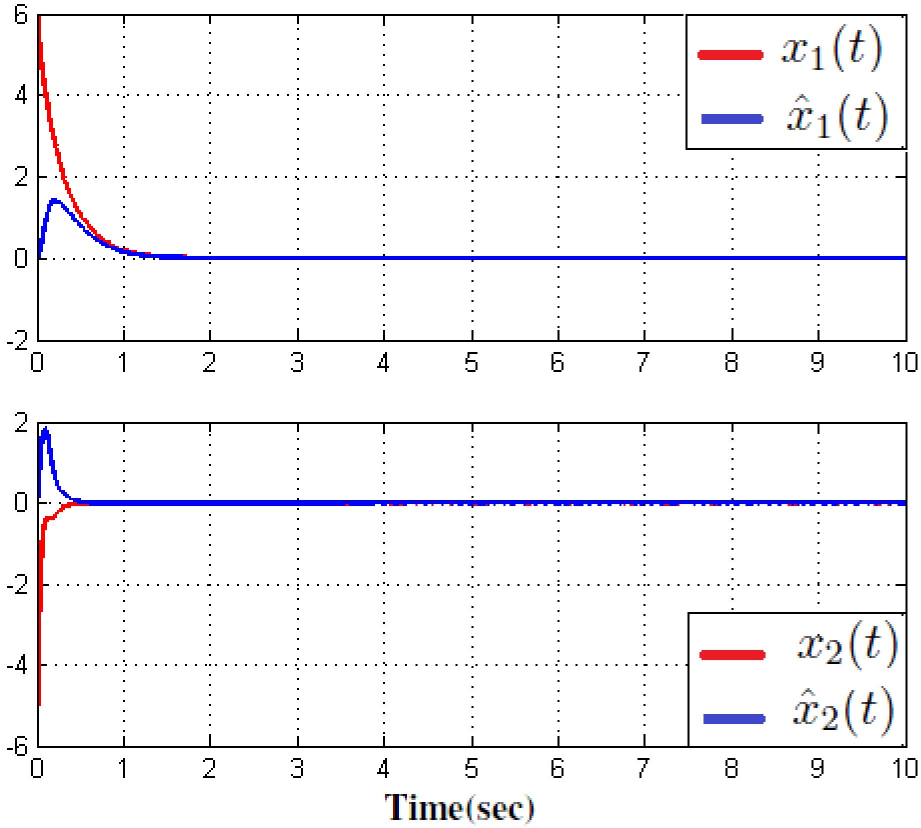

Figure 4.16: Real and estimated state response of the system

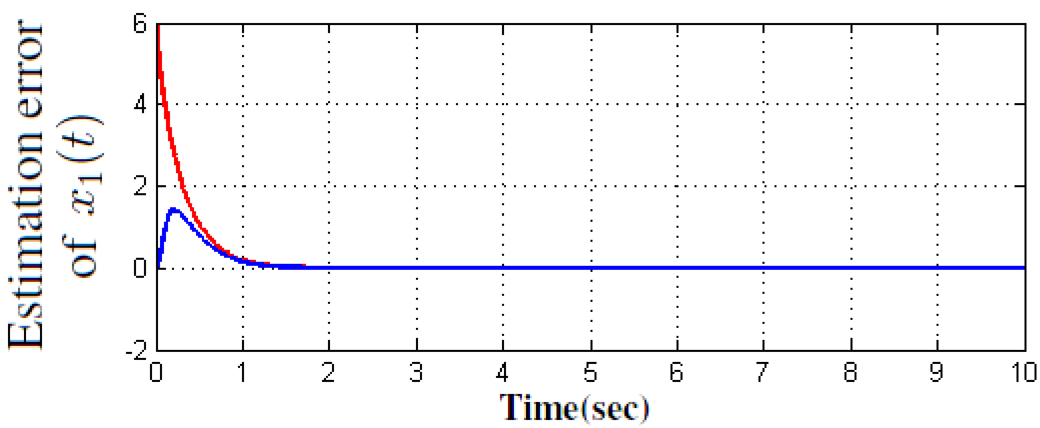

Figure 4.17: The time response error of the state $x_1(t)$

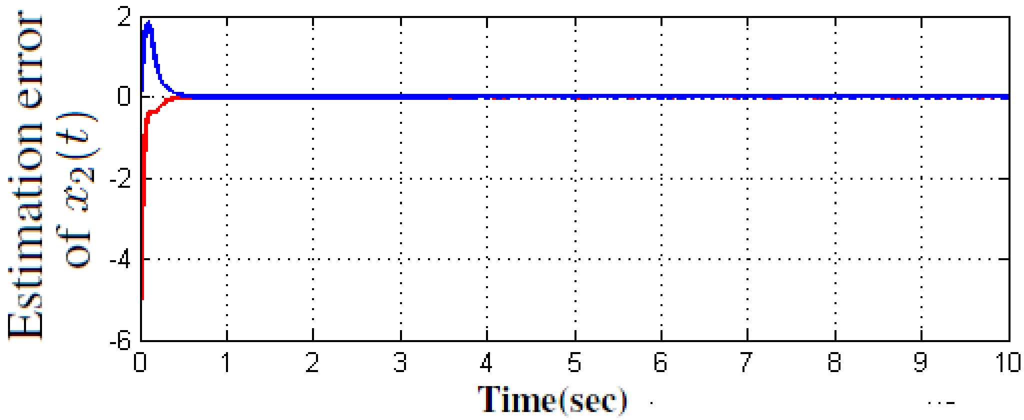

Figure 4.18: The time response error of the state $x_2(t)$

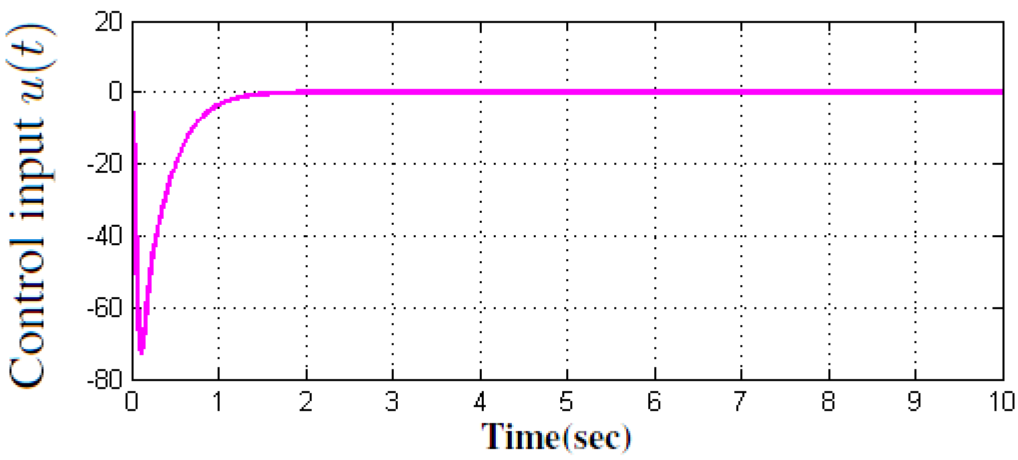

Figure 4.19: The time response of control input $u(t)$

Table 4.5: Eigenvalues of LMIs

Open loop system	Closed loop system
-7.3589,1.3589	-6.7058,-27.0841
-7.2568,1.9568	-6.7309,-26.3589
-8.3852,2.3852	-6.2263,-27.5635
-8.2377,2.9377	-6.2340,-26.8558
-8.7446,2.7446	-7.7852,-26.0046
-8.5849,3.2849	-7.8569,-25.2330
-10.5498,4.5498	-6.5515,-27.2383
-10.3456,5.0456	-6.5708,-26.5190
-5,2	-3.6155,-27.1743
-4.8137,2.5137	-3.6344,-26.4554
-6.2170,3.2170	-3.1985,-27.5914
-5.9897,3.6897	-3.2043,-26.8855
-6.6235,3.6235	-4.5358,-26.2541
-6.3866,4.0866	-4.5870,-25.5028
-8.5887,5.5887	-3.4819,-27.3080
-8.3209,6.0209	-3.4965,-26.5934

is to use second methodology that find the feed-back gain controller and observer gain in two stages.Following the procedure described in section 4.3 and theorem 4.3.2. First, by solving with respect to X_1 and Y_1 using LMI(5.22), the following matrices are obtained:
P_1= $\begin{bmatrix} 13.5519 & 2.6273 \\ 2.6273 & 7.5561 \end{bmatrix}$ and $K = [-20.2659 \quad -5.9896]$. Finally, by solving X_2 and Y_2 using LMI(5.23), we get the following matrices:

P_2 = $\begin{bmatrix} 6.4560 & 8.8903 \\ 8.8903 & 13.4783 \end{bmatrix}$
, $A = \begin{bmatrix} 129.0701 & 225.8331 \\ -69.8775 & -140.8662 \end{bmatrix}$ and L = $\begin{bmatrix} 274.7554 \\ -168.1912 \end{bmatrix}$
Simulation results are shown in figures 4.20,4.21,4.22 and 4.23. Figure 4.20 shows the real and estimated state of $x_1(t)$ and $x_2(t)$. Figure 4.21 and figure 4.22 displays the estimation errors of the states $x_1(t)$ and $x_2(t)$ respectively. They show that the estimated states converge quickly to the real states with asymptotically stable error

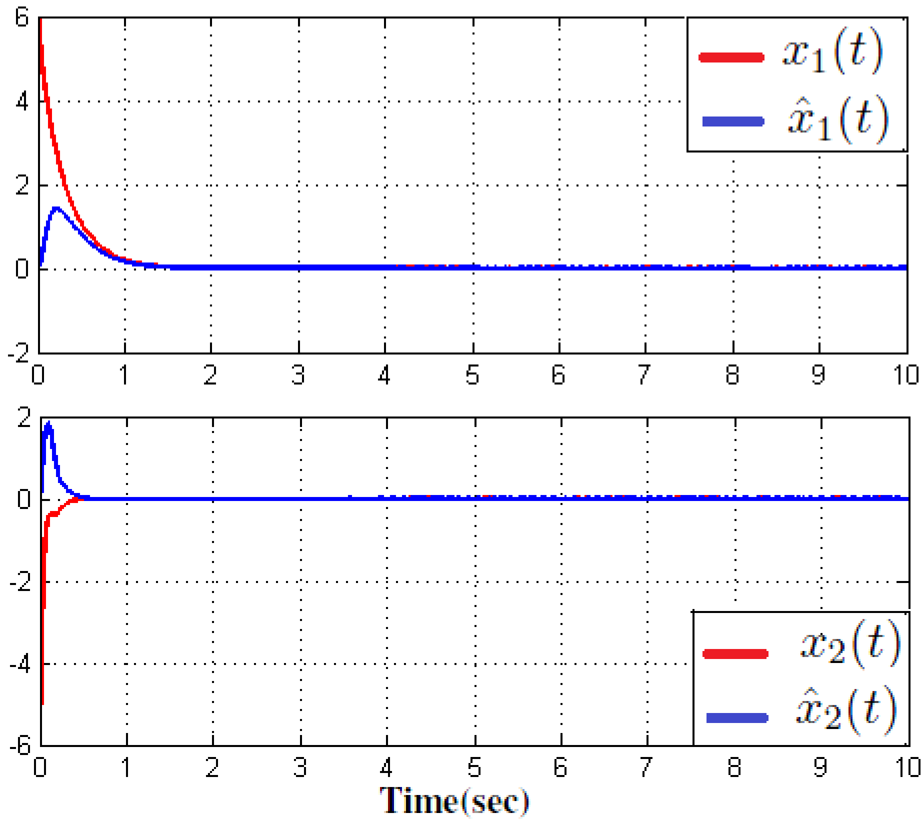

Figure 4.20: Real and estimated state response of the system

dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 4.6. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are negative and the feed-back controller stabilize the system with closed loop eigenvalue. Finally, figure 4.23 shows the input control.

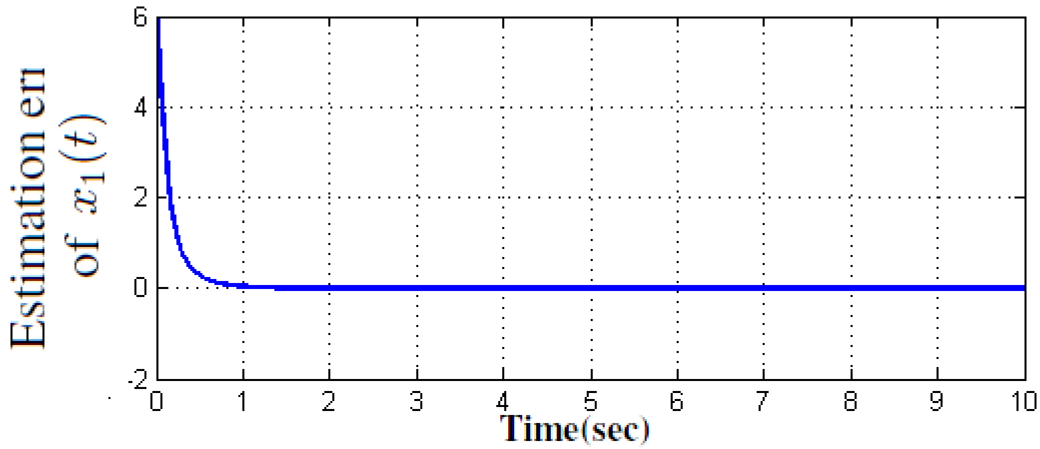

Figure 4.21: The time response error of the state $x_1(t)$

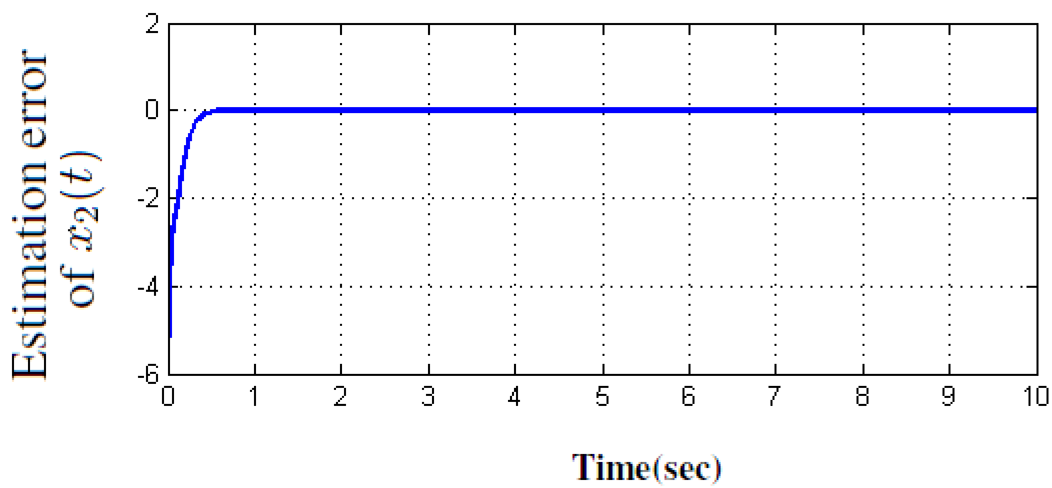

Figure 4.22: The time response error of the state $x_2(t)$

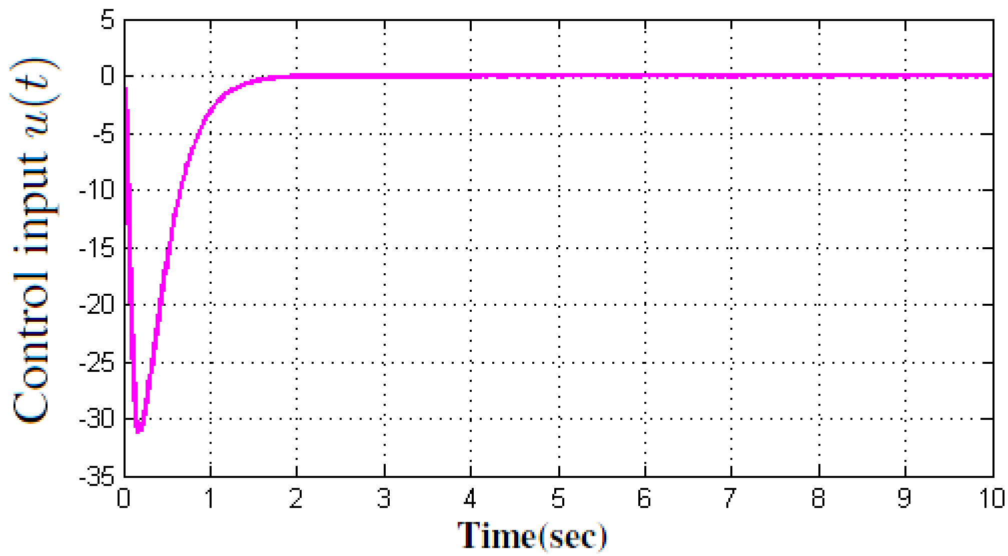

Figure 4.23: The time response of control input $u(t)$

Table 4.6: Eigenvalues of LMIs

Open loop system	Closed loop system
-7.3589,1.3589	-5.9948 ±3.2040i
-7.2568,1.9568	-5.6448 ±3.1843i
-8.3852,2.3852	-5.9948 ±0.5157i
-8.2377,2.9377	-5.6448 ±0.3738i
-8.7446,2.7446	-5.9948 ±4.9637i
-8.5849,3.2849	-5.6448 ±4.9510i
-10.5498,4.5498	-5.9948 ±0.7989i
-10.3456,5.0456	-5.6448 ±0.7156i
-5,2	-4.4948 ±2.8340i
-4.8137,2.5137	-4.1448 ±2.9926i
-6.2170,3.2170	-3.0918,-5.8978
-5.9897,3.6897	-3.1227,-5.1669
-6.6235,3.6235	-4.4948 ±4.7333i
-6.3866,4.0866	-4.1448 ±4.8299i
-8.5887,5.5887	-3.2314,-5.7582
-8.3209,6.0209	-3.3248,-4.9647

4.5 Discussion

In this chapter, the lyapunov stability theory and LMI approach are used to design of observer-based control of continuous time systems. The proposed methodology is to construct linear full order observer to guarantee the feedback controlled system is exponentially stabilizable. The results show it is fast and stabilize the system within few seconds. However, as the design of observer-based control has LMI formulation, it may be possible to get infeasibility. Thus, we suggest two methods to solve BMIs of observer-based control. Also, by using substitution method,we could avoid having equality in the LMIs.

To compare the performance of both methods, in the example 4.4.1, the states $x_3(t)$ and $x_4(t)$ of the first method reaches convergence a little faster than second method (substitution method). However, the state $x_1(t)$ of the second method reaches convergence a little faster than first method. The state $x_2(t)$ of both methods are the same. In the example 4.4.2, the states state of perturbation of the body axis forward speed $u(t)$ and pitch angle $\theta(t)$ of the first method reaches convergence a little faster than second method(substitution method). However, the state pitch rate $q(t)$ and longitudinal flapping angle $a_1(t)$ of second method reaches convergence a little faster than first method. In the example 4.4.3, the states $x_1(t)$ of the first method reaches convergence a little faster than second method(substitution method). The state $x_2(t)$ of both methods are the same.

From the eigenvalue tables, the first method has the most large negative real parts. Overall, the results indicate that the first method stabilize the system faster than second method.

Chapter 5
Design Of Observer-Based Control Of Discrete Time Systems

5.1 Introduction

The approach to design this robust observer-based control of discrete time is very important and has became a focus of much research in recent years. These motivate us to consider the observer design and robust observer-based control for uncertain systems. Recently, much effort has been devoted to design the observer-based control of discrete time of uncertain systems with many approaches. In [22], An observer-based control for a class of discrete-time non-linear systems is considered. The system non-linearity is assumed globally Lipschitz and the system is supposed to be stabilizable by a linear controller. Sufficient LMI condition is derived to ensure the stability of the considered system under the action of feedback control based on the reconstructed states. In[18],observer-based state feedback LPV controller synthesis for the nonlinear viscous Burger′s equation has been presented . The one-dimensional nonlinear Burgersequation was discretized using a finite difference scheme and the boundary conditions were taken as control inputs.A discrete-time polytopic quasi-LPV model which is affine in scheduling parameters was derived based on the reduced model, where the scheduling parameters are the reduced states. An output-feedback LPV controller with a fixed lyapunov function was designed which has a reasonable synthesis. In [29], two types of observer-based output feedback control design methods are presented, compared, and analyzed with regard to robustness to model uncertainties and insensitivity to output disturbances. An Observer design methods are proposed for two different strategies: (a) based on an observer-controller separation and (b) based on simultaneous design derived from the Finslers lemma. The controller designs are formulated in terms of LMIs that are solvable with standard software

This chapter is divided as follows. Section 5.1, presents brief an introduction. In section 5.2,problem formulation is mentioned. Section 5.3 presents the design procedure of observer-based control design of discrete time systems. In section 5.4, numerical examples are given to illustrate the effectiveness of the proposed methodology results. Finally,section 5.4 presents discussion.

5.2 Problem Formulation

The aim of this chapter is to design some state observer with output $\hat{x}$ for systems given by (2.8) in order to replace the state-feedback law by $u = K\hat{x}$. The goal of this observer design is to have a closed-loop behavior as resembling as possible to the ideal state-feedback [36]. A dynamic observer-based controller for the system (2.8) is given by

$$\hat{x}_{k+1} = Ax_k + Bu_k + L(\hat{y}_k - y_k) \tag{5.1a}$$

$$\hat{y}_k = C\hat{x}_k \tag{5.1b}$$

$$u_k = K\hat{x}_k \tag{5.1c}$$

where the parameters to design are A , $\hat{x}_k \in \mathbb{R}^n$ is the estimation of x_k , $\hat{y}_k \in \mathbb{R}^m$ is the observer output, $u_k \in \mathbb{R}^r$ is input vector, $K \in \mathbb{R}^{rxn}$ is the control gain and $L \in \mathbb{R}^{nxm}$ is the observer gain.Figure 5.1, shows the block diagram of observer-based control for LPV systems.The order of the plant and controller are the same. The number of state variables for the closed-loop system is thus double that of the open-loop plant.

5.3 Design Procedure

In this section, we consider the design of observer-based control of continuous time. The procedure is as follow. First, we find the lyapunov function for closed-loop system .Then,we take derivative of lyapunov function and we apply the lyapunov

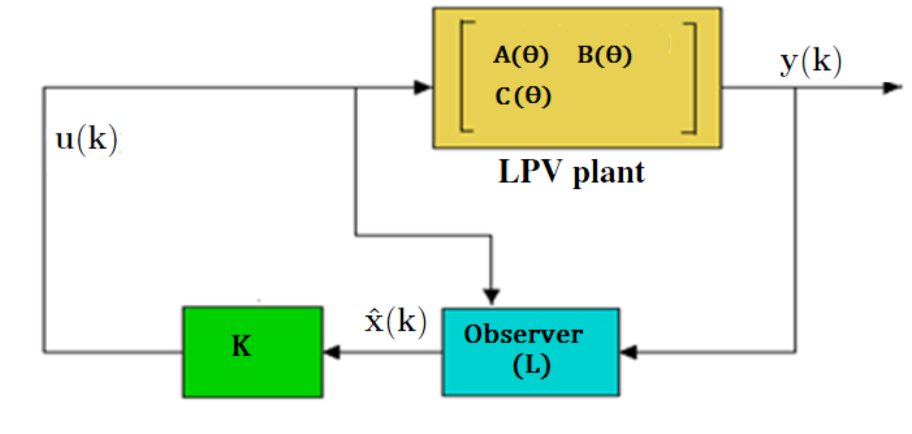

Figure 5.1: Block diagram of observer-based control for LPV systems

stability conditions on it. Finally,the control and observer gains are formulated as the bilinear matrix inequalities (BMIs) formulations which has to be converted to LMIs since the BMI is nonconvex optimization problems. According to (5.1) the dynamic of observer-based control of discrete systems is given by

$$\xi_{k+1} = \begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 + LC \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} \tag{5.2}$$

where $\xi = \begin{bmatrix} x_k \\ e_k \end{bmatrix}$. Now, the lyapunov function is given by

$$V(\xi_k) = \xi_k^T P \xi_k \tag{5.3}$$
$$V(\xi_{k+1}) = \xi_{k+1}^T P \xi_{k+1}$$

Then,

$$V(\xi_{k+1}) - V(\xi_k) = \xi_{k+1}^T P \xi_{k+1} - \xi_k^T P \xi_k \tag{5.4}$$

Then, we substitute the equation (5.2) into equation (5.4) as follow:

$$V(\xi_{k+1}) - V(\xi_k) = \begin{bmatrix} x_k \\ e_k \end{bmatrix}^T \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix}^T P \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} - \begin{bmatrix} x_k \\ e_k \end{bmatrix}^T P \begin{bmatrix} x_k \\ e_k \end{bmatrix} \quad (5.5)$$

$$V(\xi_{k+1}) - V(\xi_k) = \begin{bmatrix} x_k \\ e_k \end{bmatrix}^T \Bigg(\begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix}^T P \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} - P \Bigg) \begin{bmatrix} x_k \\ e_k \end{bmatrix} \quad (5.6)$$

System (5.6) is asymptotically stable if and only if there is matrix $P > 0$ such that

$$\begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix}^T P \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} - P < 0 \quad (5.7)$$

$$P - \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix}^T P \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} > 0 \quad (5.8)$$

Let matrix P to be symmetric positive definite such as $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$. Then, we substitute the matrix P in (5.8) we get

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A(\theta) + BK & -BK \\ A(\theta) - A_0 & A_0 - LC \end{bmatrix} > 0 \quad (5.9)$$

Then, (5.9) is equivalent by Schur complement to the following inequality:

$$\begin{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} & \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} & \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} > 0 \tag{5.10}$$

Or equivalently,

$$\begin{bmatrix} P_1 & 0 & A(\theta)^T P_1 + K^T B^T P_1 & A(\theta)^T P_2 - A_0^T P_2 \\ 0 & P_2 & -K^T B^T P_1 & A_0^T P_2 - C^T L^T P_2 \\ P_1 A(\theta) + P_1 BK & -PBK & P_1 & 0 \\ P_2 A(\theta) - P_2 A_0 & P_2 A_0 - P_2 LC & 0 & P_2 \end{bmatrix} > 0 \tag{5.11}$$

Inequality (5.11) is BMI which is nonconvex optimization problems. The problems formulated in BMI terms may have multiple local solutions and cannot be resolved using the convex optimization techniques developed for solving LMI problems, as they are often difficult nonconvex problems. Some heuristic methods which enable solutions to these types of problems have been proposed We can solve this type of problem using linearization technique by a change in variables. The product of two variables is replaced by a new variable as follows:

Let $P_1 B = BZ$ and $K = Z^{-1}G$ then $P_1 BK = BZ^{-1}ZG = BG$. Similarly, $K^TB^TP_1 = G^TB^T$. Other, $X = P_2A_0$ and $X^T = A_0^TP_2$, $Y = P_2L$ and $Y^T = L^TP_2$ where $X \in \mathbb{R}^{nxn}$, $G \in \mathbb{R}^{rxn}$, $Y \in \mathbb{R}^{nxm}$ and Z is a scaler. The result:

$$\begin{bmatrix} P_1 & 0 & A(\theta)^TP_1 + G^TB^T & A(\theta)^TP_2 - X^T \\ 0 & P_2 & -G^TB^T & X^T - C^TY^T \\ P_1A(\theta) + BG & -BG & P_1 & 0 \\ P_2A(\theta) - X & X - YC & 0 & P_2 \end{bmatrix} > 0 \quad (5.12)$$

Theorem 5.3.1:*System (2.8) is exponentially stabilizable by the feedback $u = k\hat{x}$ if there exist symmetric matrices $P_1 \in \mathbb{R}^{nxn}$ and $P_2 \in \mathbb{R}^{nxn}$,matrices $X \in \mathbb{R}^{nxn}$, $G \in \mathbb{R}^{rxn}$, $Y \in \mathbb{R}^{nxm}$ and Z is a scaler such that*

$$P_1 > 0 \quad (5.13a)$$

$$P_2 > 0 \quad (5.13b)$$

$$P_1B = BZ \quad (5.13c)$$

$$\begin{bmatrix} P_1 & 0 & A(\theta)^TP_1 + G^TB^T & A(\theta)^TP_2 - X^T \\ 0 & P_2 & -G^TB^T & X^T - C^TY^T \\ P_1A(\theta) + BG & -BG & P_1 & 0 \\ P_2A(\theta) - X & X - YC & 0 & P_2 \end{bmatrix} > 0 \quad (5.13d)$$

Then, the stabilizing observer-based control gain is given by $K = Z^{-1}G$ and observer gain $L = P_2^{-1}Y$ and $A_0 = P_2^{-1}X$.

The second method to solve (5.11) is to solve first $P1$ and K. Then, we solve for P_2 , A_0 and L to avoid getting equality relation in the first method. This method is called substitution method. The closed loop of systems (2.8) with state feedback control $u = Kx_k$:

$$x_{k+1} = (A(\theta_k) + BK)x_k \quad (5.14)$$

According to lyapunov function is given in (5.3) and (5.4)

$$V(\xi_k) = {x_k}^T P_1 x_k \tag{5.15a}$$

$$V(\xi_{k+1}) = {x_k}^T (A(\theta_k) + BK)^T P_1 (A(\theta_k) + BK) x_k \tag{5.15b}$$

$$V(\xi_{k+1}) - V(\xi_k) = {x_k}^T (A(\theta_k) + BK)^T P_1 (A(\theta_k) + BK) x_k - {x_k}^T P_1 x_k \tag{5.16}$$

$$V(\xi_{k+1}) - V(\xi_k) = {x_k}^T ((A(\theta_k) + BK)^T P_1 (A(\theta_k) + BK) - P_1) x_k \tag{5.17}$$

The system (5.17) is asymptotically stable if and only if there is symmetric matrix $P_1 > 0$ such that

$$(A(\theta_k) + BK)^T P_1 (A(\theta_k) + BK) - P_1 < 0 \tag{5.18}$$

$$P_1 - (A(\theta_k) + BK)^T P_1 (A(\theta_k) + BK) > 0 \tag{5.19}$$

Then, (5.19) is equivalent by Schur complement to the following inequality:

$$\begin{bmatrix} P_1 & A(\theta_k)^T P_1 + K^T B^T P_1 \\ P_1 A(\theta_k) + P_1 BK & P_1 \end{bmatrix} > 0 \tag{5.20}$$

The inequality (5.20) is BMI which is nonconvex optimization problems can be transformed into LMI by using change variable , the product of two variables is replaced by a new variable. Let $K = Y_1 P_1$ and $X_1 = P^{-1}$. The result

$$\begin{bmatrix} X_1 & X_1 A(\theta_k)^T + Y_1^T B^T \\ A(\theta_k) X_1 + BY_1 & X_1 \end{bmatrix} > 0 \tag{5.21}$$

By solving with respect to X_1 and Y_1, we get K and P_1. To find P_2,A_0 and L,the inequality (5.9) could be written as:

$$\begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} < 0 \tag{5.22}$$

$$\begin{bmatrix} A(\theta)^T+K^TB^T & A(\theta)^T-A_0^T \\ -K^TB^T & A_0^T-C^TL^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P2 \end{bmatrix} \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & P2 \end{bmatrix} < 0 \tag{5.23}$$

$$\begin{bmatrix} A(\theta)^TP_1+K^TB^TP_1 & A(\theta)^TP_2-A_0^TP_2 \\ -K^TB^TP_1 & A_0^TP_2-C^TL^TP_2 \end{bmatrix} \begin{bmatrix} A(\theta)+BK & -BK \\ A(\theta)-A_0 & A_0-LC \end{bmatrix} - \begin{bmatrix} P_1 & 0 \\ 0 & P2 \end{bmatrix} < 0 \tag{5.24}$$

$$\begin{bmatrix} (A(\theta)^TP_1+K^TB^TP_1)(A(\theta)+BK)+(A(\theta)^TP_2-A_0^TP_2)(A(\theta)-A_0)-P_1 \\ (-K^TP^TP_1)(A(\theta)+BK)+(A_0^TP_2-C^TL^TP_2)(A(\theta)-A_0) \end{bmatrix.$$

$$\begin{matrix} (A(\theta)P_1+K^TB^TP_1)(-BK)+(A(\theta)^TP_2-A_0^TP_2)(A_0-LC) \\ (-K^TB^TP_1)(-BK)+(A_0^TP_2-C^TL^TP_2)(A_0-LC)-P_2 \end{matrix} \Bigg] < 0 \tag{5.25}$$

$$
\begin{bmatrix}
\begin{gathered}A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK \\ +A(\theta)^T P_2 A(\theta) - A(\theta)^T P_2 A_0 - A_0^T P_2 A(\theta) + A_0^T P_2 A_0 - P_1\end{gathered} & \begin{gathered}-A(\theta)^T P_1 BK - K^T B^T P_1 BK + A(\theta)^T P_2 A_0 \\ -A(\theta)^T P_2 LC - A_0^T P_2 A_0 + A_0^T P_2 LC\end{gathered} \\
\begin{gathered}-K^T B^T P_1 A(\theta) - K^T B^T P_1 BK + A_0^T P_2 A(\theta) - A_0^T P_2 A_0 \\ -C^T L^T P_2 A(\theta) + C^T L^T P_2 A_0\end{gathered} & \begin{gathered}K^T B^T P_1 BK + A_0^T P_2 A_0 - A_0^T P_2 LC \\ -C^T L^T P_2 A_0 + C^T L^T P_2 LC - P_2\end{gathered}
\end{bmatrix} < 0 \tag{5.26}
$$

Then,the inequality (5.26) is equivalent by schur complement to the following inequality:

$$
\begin{bmatrix}
\begin{gathered}A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK \\ +A(\theta)^T P_2 A(\theta) - A(\theta)^T P_2 A_0 - A_0^T P_2 A(\theta) - P_1\end{gathered} & \begin{gathered}-A(\theta)^T P_1 BK - K^T B^T P_1 BK + A(\theta)^T P_2 A_0 \\ -A(\theta)^T P_2 LC - A_0^T P_2 A_0 + A_0^T P_2 LC\end{gathered} & A_0^T P_2 \\
\begin{gathered}-K^T B^T P_1 A(\theta) - K^T B^T P_1 BK + A_0^T P_2 A(\theta) - A_0^T P_2 A_0 \\ -C^T L^T P_2 A(\theta) + C^T L^T P_2 A_0\end{gathered} & \begin{gathered}K^T B^T P_1 BK + A_0^T P_2 A_0 - A_0^T P_2 LC \\ -C^T L^T P_2 A_0 + C^T L^T P_2 LC - P_2\end{gathered} & 0 \\
P_2 A_0 & 0 & -P_2
\end{bmatrix} < 0 \tag{5.27}
$$

According to property of LMI, the block diagonal matrices could be considered as separate constraint.

$$
\begin{aligned}
&A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK + \\
&\qquad A(\theta)^T P_2 A(\theta) - A(\theta)^T P_2 A_0 - A_0^T P_2 A(\theta) - P_1 < 0
\end{aligned} \tag{5.28}
$$

By using linearization technique by a change in variables. The product of two variables $P_2 A_0$ is replaced by a new variable Was follows:

$$
\begin{aligned}
&A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK + \\
&\qquad A(\theta)^T P_2 A(\theta) - A(\theta)^T W - W^T A(\theta) - P_1 < 0
\end{aligned} \tag{5.29}
$$

By solving(5.29) with respect to W, we get A_0 and P_2

Also, the (5.26) is equivalent by schur complement to the following inequality:

$$
\begin{bmatrix}
\begin{matrix} A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK \\ +A(\theta)^T P_2 A(\theta) - A(\theta)^T P_2 A_0 - A_0^T P_2 A(\theta) + A_0^T P_2 A_0 - P_1 \end{matrix} & \begin{matrix} -A(\theta)^T P_1 BK - K^T B^T P_1 BK + A(\theta)^T P_2 A_0 \\ -A(\theta)^T P_2 LC - A_0^T P_2 A_0 + A_0^T P_2 LC \end{matrix} & 0 \\
\begin{matrix} -K^T B^T P_1 A(\theta) - K^T B^T P_1 BK + A_0^T P_2 A(\theta) - A_0^T P_2 A_0 \\ -C^T L^T P_2 A(\theta) + C^T L^T P_2 A_0 \end{matrix} & \begin{matrix} K^T B^T P_1 BK + A_0^T P_2 A_0 - A_0^T P_2 LC \\ -C^T L^T P_2 A_0 - P_2 \end{matrix} & C^T L^T P_2 \\
0 & P_2 LC & -P_2
\end{bmatrix} < 0 \tag{5.30}
$$

By solving(5.30), we get the observer gain L.

Theorem 5.3.2: *System (2.8) is exponentially stabilizable by the feedback $u = k\hat{x}$:*

(1) If there exist symmetric matrix $X_1 \in \mathbb{R}^{nxn}$ and matrix $Y_1 \in \mathbb{R}^{rxn}$ such that

$$X_1 > 0 \tag{5.31a}$$

$$\begin{bmatrix} X_1 & X_1 A(\theta_k)^T + Y_1^T B^T \\ A(\theta_k) X_1 + B Y_1 & X_1 \end{bmatrix} > 0 \tag{5.31b}$$

Then, the control gain is given by $K = Y_1 P_1$ and $P_1 = X^{-1}$

(2) If there exist symmetric matrix $P_2 \in \mathbb{R}^{nxn}$,matrix $W \in \mathbb{R}^{nxn}$ such that

$$P_2 > 0 \tag{5.32a}$$

$$\begin{aligned} A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK + \\ A(\theta)^T P_2 A(\theta) - A(\theta)^T W - W^T A(\theta) - P_1 < 0 \end{aligned} \tag{5.32b}$$

Then, $A_0 = P_2^{-1} W$.

(3) If there exist symmetric matrix $L \in \mathbb{R}^{mxn}$ such that

$$\begin{bmatrix} \begin{matrix} A(\theta)^T P_1 A(\theta) + A(\theta)^T P_1 BK + K^T B^T P_1 A(\theta) + K^T B^T P_1 BK \\ +A(\theta)^T P_2 A(\theta) - A(\theta)^T P_2 A_0 - A_0^T P_2 A(\theta) + A_0^T P_2 A_0 - P_1 \end{matrix} & \begin{matrix} -A(\theta)^T P_1 BK - K^T B^T P_1 BK + A(\theta)^T P_2 A_0 \\ -A(\theta)^T P_2 LC - A_0^T P_2 A_0 + A_0^T P_2 LC \end{matrix} & 0 \\ \begin{matrix} -K^T B^T P_1 A(\theta) - K^T B^T P_1 BK + A_0^T P_2 A(\theta) - A_0^T P_2 A_0 \\ -C^T L^T P_2 A(\theta) + C^T L^T P_2 A_0 \end{matrix} & \begin{matrix} K^T B^T P_1 BK + A_0^T P_2 A_0 - A_0^T P_2 LC \\ -C^T L^T P_2 A_0 - P_2 \end{matrix} & C^T L^T P_2 \\ 0 & P_2 LC & -P_2 \end{bmatrix} < 0 \tag{5.33}$$

By solving (5.33), we get the observer gain L.

5.4 Numerical Examples

In the previous section,the design of observer-based control of discrete systems was presented using LMIs and lyapunov function. The design will be investigated by two examples using the simulation of the software MATLAB using the YALMIP toolbox and the solver SeDuMi. Section 5.4.1 presents example of simulation of observer-based control of discrete system with three uncertain parameters. Finally, section 5.4.2,presents example of simulation of observer-based control of discrete system with all elements of system matrix are uncertain parameters.

5.4.1 First Example

Consider a discrete time LPV system :

$$x_{k+1} = \begin{bmatrix} a_k & b_k & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & c_k & 0 & -0.7 \end{bmatrix} x_k + \begin{bmatrix} 1.1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u_k,\ y_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} x_k \quad (5.34)$$

with three independent scalar uncertainties $a_k \in [0.2, 0.8]$,$b_k \in [0.2, 0.7]$ and $c_k \in [0.1, 1.5]$. This system is trivially a polytopic model with i = 8 vertices obtained by taking combinations of extremal values of the uncertainties. The following are the vertices A_i which are not stable

$$A_1 = \begin{bmatrix} A1 = 0.2 & 0.2 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 0.1 & 0 & -0.7 \end{bmatrix}, A_2 = \begin{bmatrix} A2 = 0.2 & 0.2 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 1.5 & 0 & -0.7 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.2 & 0.7 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 0.1 & 0 & -0.7 \end{bmatrix}, A_4 = \begin{bmatrix} 0.2 & 0.7 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 1.5 & 0 & -0.7 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0.8 & 0.2 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 0.1 & 0 & -0.7 \end{bmatrix}, A_6 = \begin{bmatrix} 0.8 & 0.2 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 1.5 & 0 & -0.7 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0.8 & 0.7 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 0.1 & 0 & -0.7 \end{bmatrix} \text{ and } A_8 = \begin{bmatrix} 0.8 & 0.7 & -2 & 0 \\ 0.5 & 0 & 1 & 0 \\ -0.8 & 0 & -0.3 & 0 \\ 1 & 0.5 & 0 & -0.7 \end{bmatrix}$$

Following the procedure described in section 5.3 and theorem 5.3.1, the following matrices are obtained:

$$P_1 = \begin{bmatrix} 0.4634 & 0 & 0 & 0 \\ 0 & 0.4634 & 0 & 0 \\ 0 & -0.0000 & 0.1371 & 0.0259 \\ 0 & 0.0000 & 0.0259 & 0.0221 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 1.3029 & 0.4889 & 0.4046 & -0.0116 \\ 0.4889 & 1.5602 & 0.5160 & -0.0303 \\ 0.4046 & 0.5160 & 3.8080 & 0.0355 \\ -0.0116 & -0.0303 & 0.0355 & 0.0601 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.4969 & 0.4096 & -2.0297 & -0.0144 \\ 0.4583 & 0.0443 & 0.9809 & 0.0233 \\ -0.7268 & 0.0364 & -0.3355 & 0.0024 \\ 0.7781 & 0.9209 & 0.1674 & -0.6600 \end{bmatrix}$$, observer gains will be given by

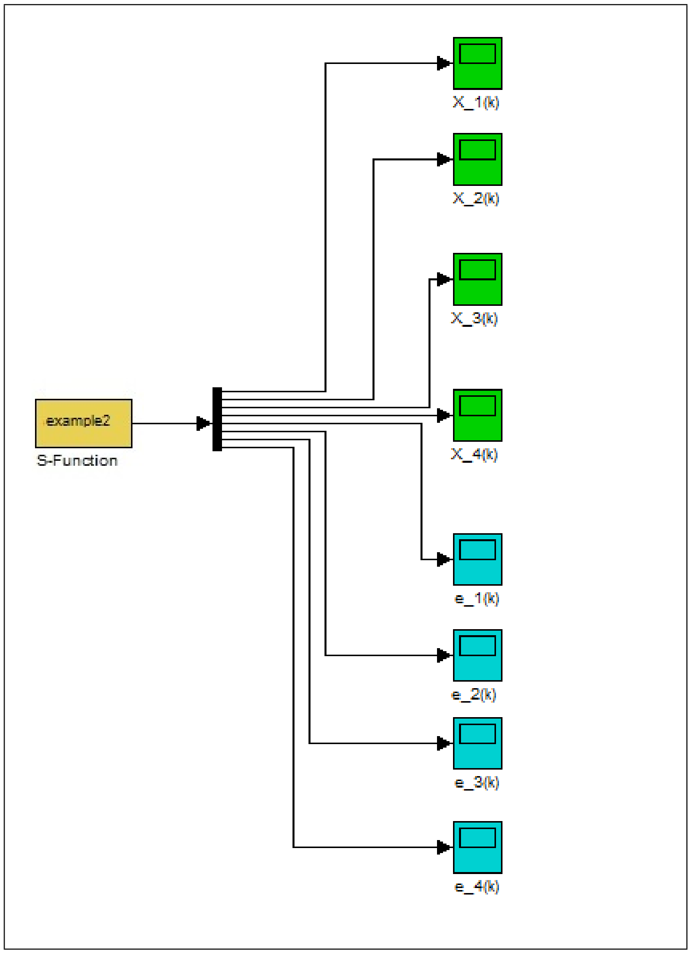

Figure 5.2: Simulation of discrete system using S-function in simulink

$L = \begin{bmatrix} 0.4741 & -2.8854 \\ 0.4494 & 0.9900 \\ -0.7553 & -0.2736 \\ 0.4596 & 0.2259 \end{bmatrix}$ and the robust state-feedback gain controller

$K = \begin{bmatrix} -0.4173 & -0.3721 & 1.7398 & -0.0018 \\ -0.5055 & 0.0188 & -1.0187 & 0.0024 \end{bmatrix}$. The simulation done for the sys

tem using S-function in simulink as shown in the figure 5.2.

Simulation results are shown in Figures 5.3, 5.4 and 5.5. Figure 5.3 shows the

Table 5.1: Eigenvalues of LMIs

Open loop system	Closed loop system
-0.7000,1.2374, 0.0744,-1.4118	0.0063 ±0.0711i,-0.557
-0.7000,1.2374, 0.0744,-1.4118	0.0086 ±0.0707i,-0.558
-0.7000,1.2318,0.2356,-1.5675	0.0834,-0.0988,-0.5357, -0.6892
-0.7000,1.2318,0.2356,-1.5675	0.0854,-0.0954,-0.5371,-0.6932
-0.7000,1.6367,0.0660,-1.2028	0.4253,0.0366,-0.4034,-0.6988
-0.7000,1.6367,0.0660,-1.2028	0.4252,0.0414,-0.4036,-0.7033
-0.7000,1.6563,0.2022,-1.3585	0.4434,-0.0054,-0.3814,-0.6969
-0.7000,1.6563,0.2022,-1.3585	0.4434,-0.0039,-0.3816,-0.6982

real and estimated state of $x_1(k)$,$x_2(k)$,$x_3(k)$ and $x_4(k)$ and figure 5.4 displays the estimation errors of the states $x_1(k)$,$x_2(k)$,$x_3(k)$ and $x_4(k)$. They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics.The system eigenvalues with and without the proposed observer-based control are given in table 5.1. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are inside the unit circle and the feedback controller stabilize the system with closed loop eigenvalue. Finally, Figure 5.5 shows the input control $u_1(t)$ and $u_2(t)$.

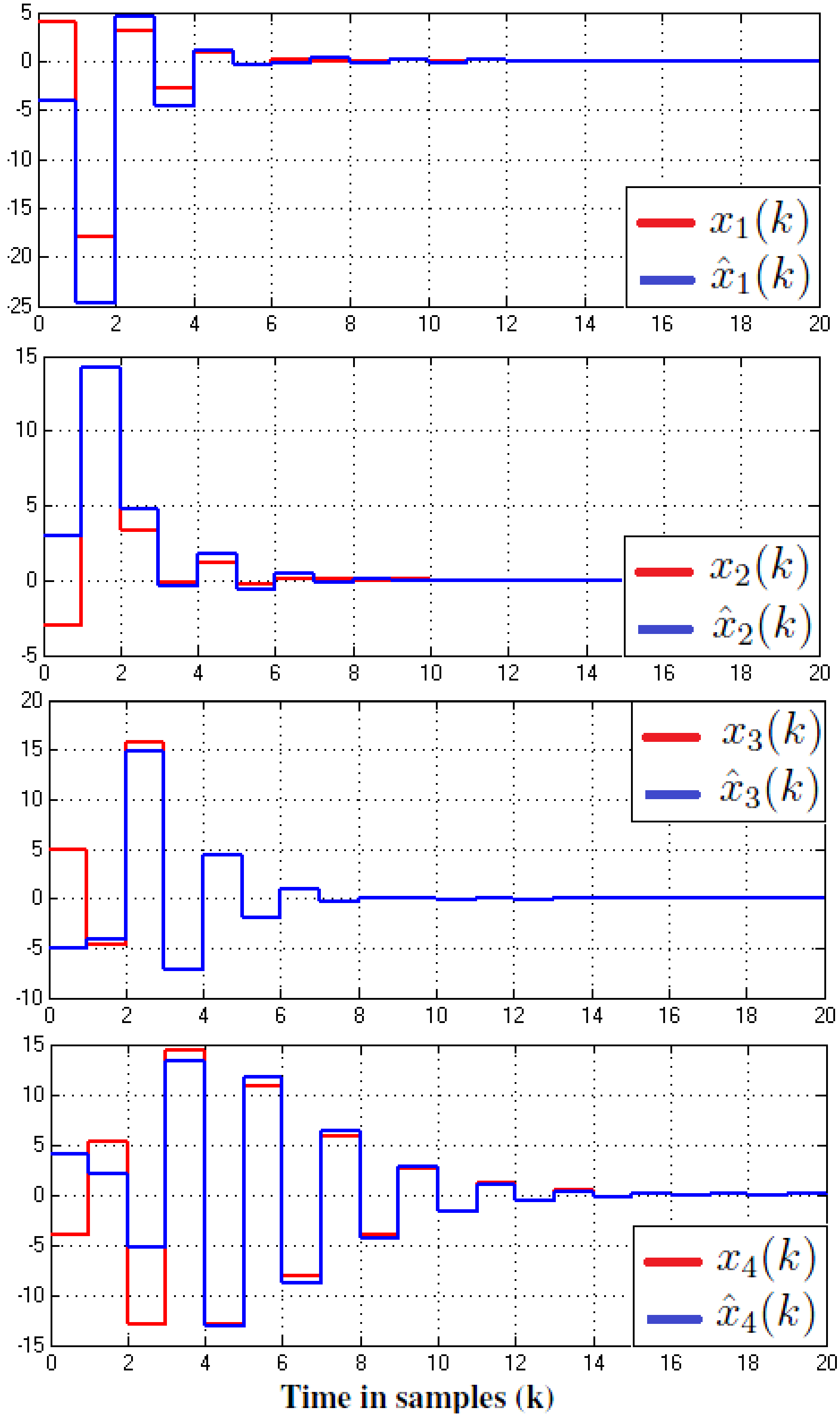

Figure 5.3: Real and estimated state response of the system

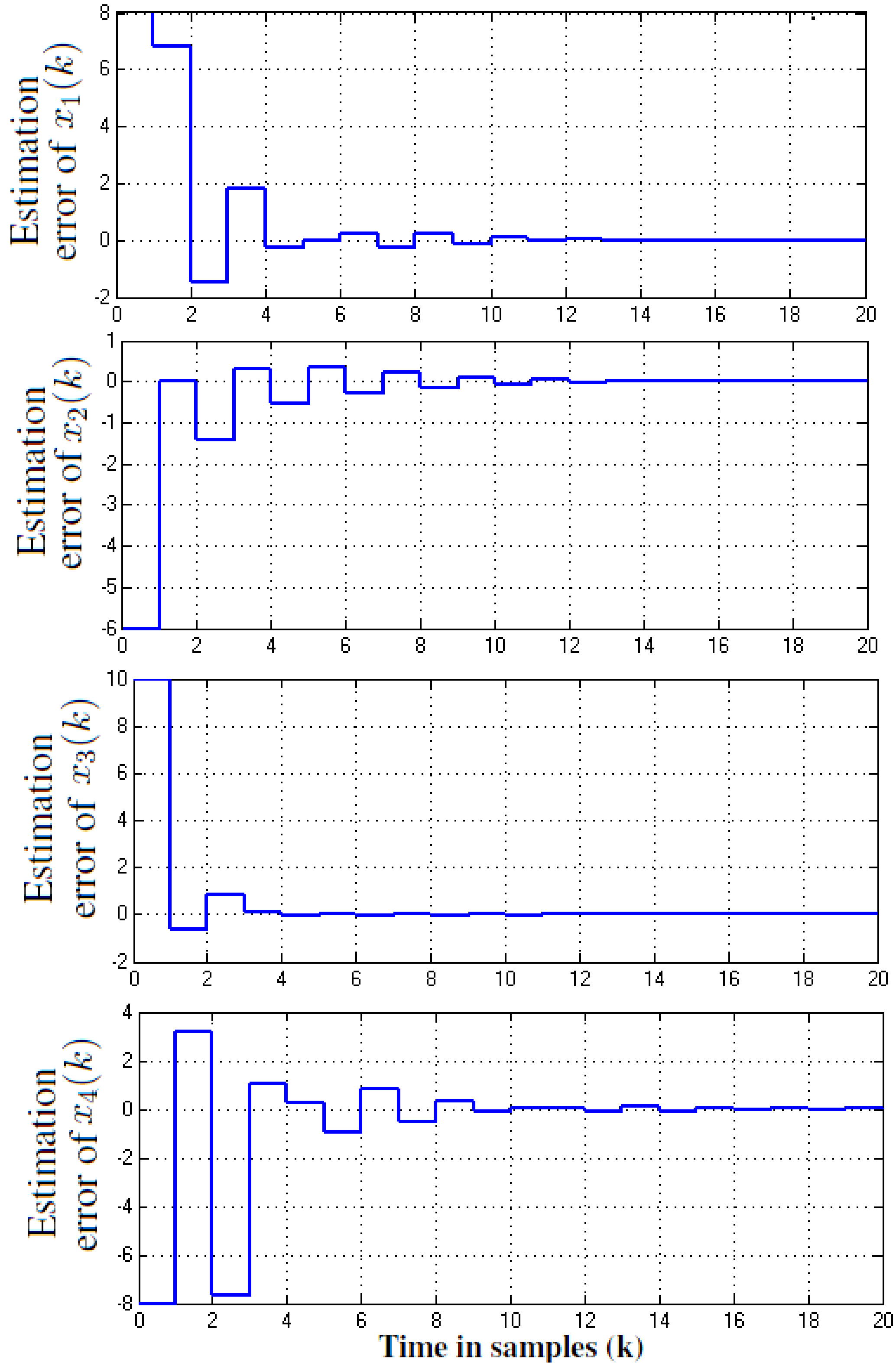

Figure 5.4: The time response error of the system

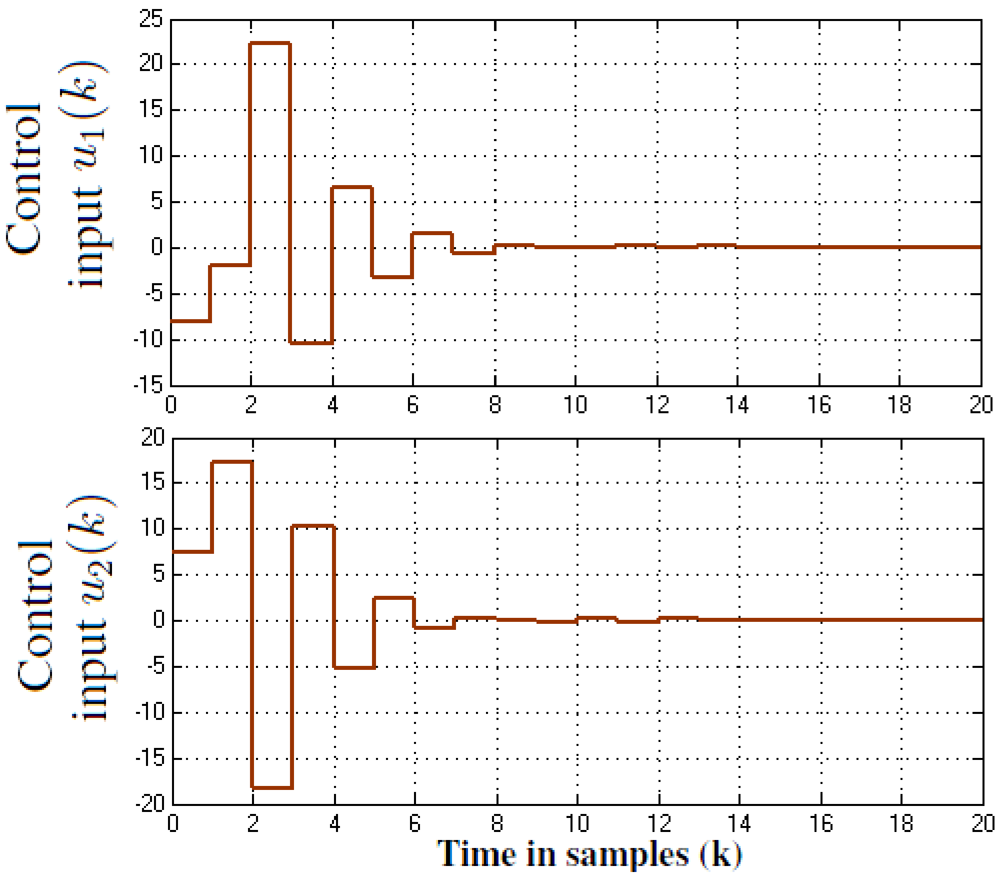

Figure 5.5: The time response of control input $u_1(t)$ and $u_2(t)$

5.4.2 Second Example

Consider a discrete time LPV system :

$$x_{k+1} = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k, y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k \tag{5.35}$$

with four uncertainties parameters a_k, b_k, c_k and d_k. As shown, all system matrix elements are uncertain parameters. Therefore, the range of parameters as follow $a_k \in [0.2, 0.4]$, $b_k \in [0.9, 1.1]$, $c_k \in [0.9, 1.1]$ and $d_k \in [0, 0.1]$. This system is trivially a polytopic model with i = 16 vertices obtained by taking combinations of extremal values of the uncertainties. The following are the vertices A_i which are not stable

$$A_1 = \begin{bmatrix} 0.2 & 0.9 \\ 0.9 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0.2 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0.2 & 0.9 \\ 1.1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0.2 & 0.9 \\ 1.1 & 0.1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0.2 & 1.1 \\ 0.9 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0.2 & 1.1 \\ 0.9 & 0.1 \end{bmatrix}, A_7 = \begin{bmatrix} 0.2 & 1.1 \\ 1.1 & 0 \end{bmatrix} \text{ and } A_8 = \begin{bmatrix} 0.2 & 1.1 \\ 1.1 & 0.1 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0.4 & 0.9 \\ 0.9 & 0 \end{bmatrix}, A_{10} = \begin{bmatrix} 0.4 & 0.9 \\ 0.9 & 0.1 \end{bmatrix}, A_{11} = \begin{bmatrix} 0.4 & 0.9 \\ 1.1 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.4 & 0.9 \\ 1.1 & 0.1 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0.4 & 1.1 \\ 0.9 & 0 \end{bmatrix}, A_{14} = \begin{bmatrix} 0.4 & 1.1 \\ 0.9 & 0.1 \end{bmatrix}, A_{15} = \begin{bmatrix} 0.4 & 1.1 \\ 1.1 & 0 \end{bmatrix} \text{ and } A_{16} = \begin{bmatrix} 0.4 & 1.1 \\ 1.1 & 0.1 \end{bmatrix}$$

Following the procedure described in section 5.3 and theorem 5.3.1, the following matrices are obtained:

$$P_1 = \begin{bmatrix} 0.9486 & 0 \\ 0 & 0.5020 \end{bmatrix}, P_2 = \begin{bmatrix} 1.2111 & 0.1359 \\ 0.1359 & 1.3082 \end{bmatrix}$$

$A = \begin{bmatrix} 0.3012 & 1.0774 \\ 0.9719 & 0.0557 \end{bmatrix}$, observer gains will be given by $L = \begin{bmatrix} 0.3063 & 1.0839 \\ 0.9780 & 0.0610 \end{bmatrix}$ and the robust state-feedback gain controller $K = [-0.2555 \quad -0.7589]$.

Table 5.2: Eigenvalues of LMIs

Open loop system	Closed loop system
-0.8055,1.0055	-0.3852, 0.3297
-0.7514,1.0514	-0.3425,0.3870
1.1000,-0.9000	-0.4227,0.3672
1.1462,-0.8462	-0.3793,0.4238
1.1000,-0.9000	-0.5825, 0.5270
1.1462,-0.8462	-0.5372, 0.5817
-1.0045,1.2045	-0.6409, 0.5854
-0.9511,1.2511	-0.5952,0.6397
-0.7220,1.1220	0.4359,-0.2913
-0.6624,1.1624	0.4793,-0.2348
1.2149,-0.8149	0.4728,-0.3283
1.2562,-0.7562	0.5168,-0.2723
1.2149,-0.8149	0.6310,-0.4865
1.2562, -0.7562	0.6768, -0.4322
-0.9180, 1.3180	0.6890, -0.5445
-0.8602, 1.3602	0.7352, -0.4907

Simulation results are shown in figures 5.6,5.7, 5.8 and 5.9. Figure 5.6 shows the real and estimated state of $x_1(k)$ and $x_2(k)$. Figure 5.7 and figure 5.8 displays the estimation errors of the states $x_1(k)$ and $x_2(k)$ respectively . They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics. The system eigenvalues with and without the proposed observer-based control are given in table 5.2. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are inside the unit circle and the feedback controller stabilize the system with closed loop eigenvalue. Finally, figure 5.9 shows the input control. Another way is to use second methodology that find the feedback gain controller K and P_1.Then, we find the A_0,P_2 and observer gain L. Following the procedure described in section 5.3 and theorem 5.3.2. First, by

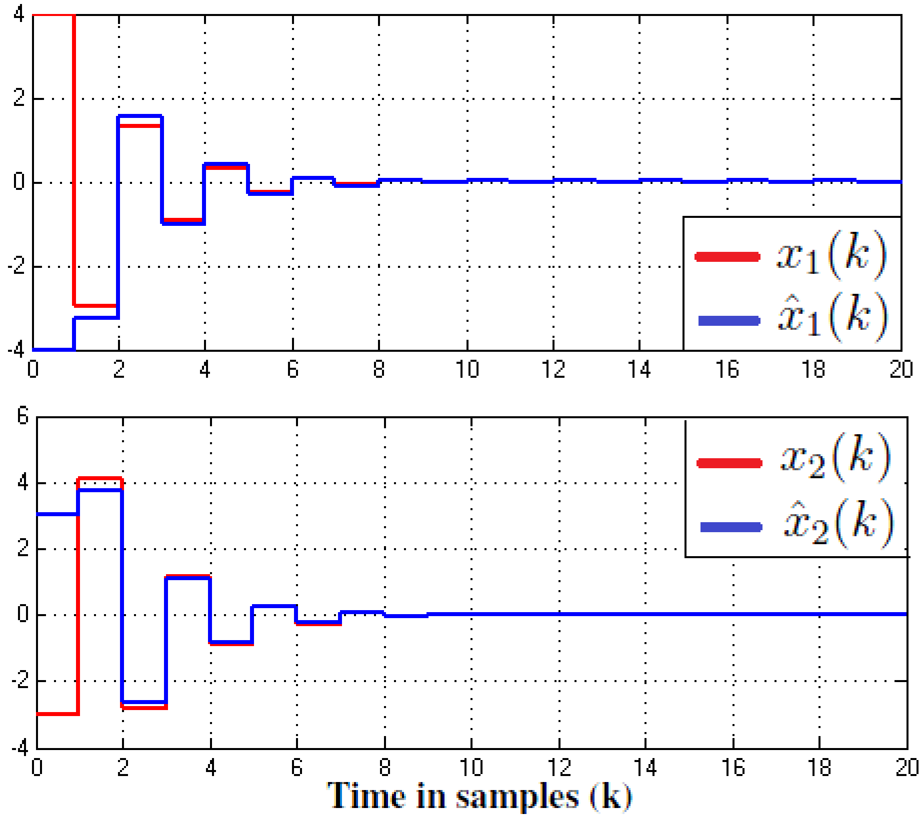

Figure 5.6: Real and estimated state response of the system

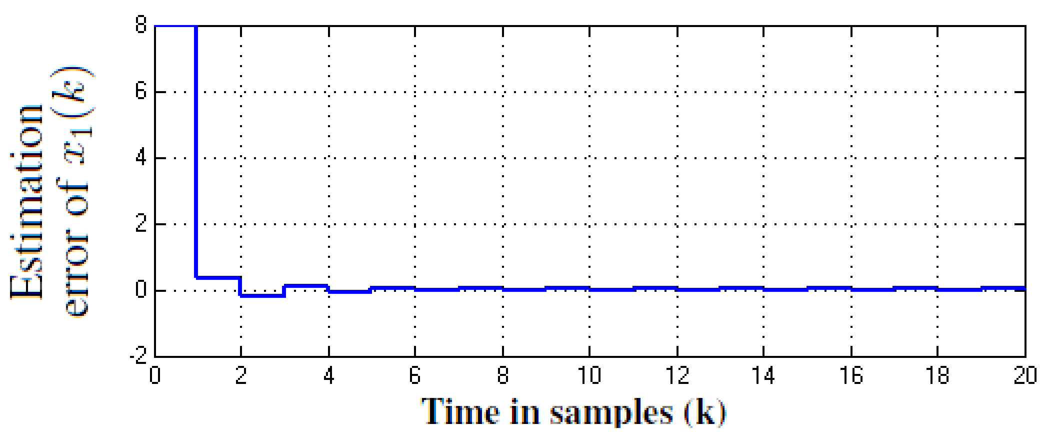

Figure 5.7: The time response error of the state $x_1(t)$

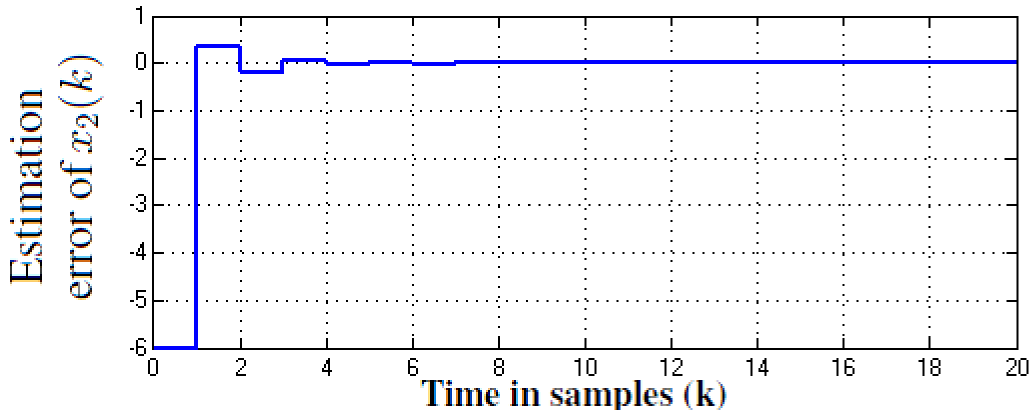

Figure 5.8: The time response error of the state $x_2(t)$

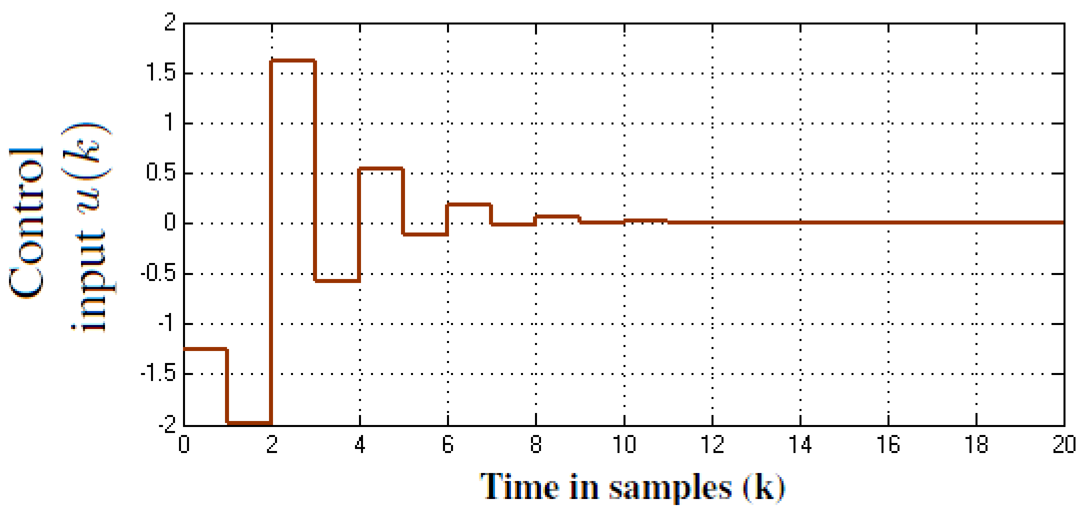

Figure 5.9: The time response of control input $u(t)$

solving with respect to X_1 and Y_1 using LMI(5.21), the following matrices are obtained:

P_1= $\begin{bmatrix} 1.6284 & 0.0310 \\ 0.0310 & 0.7233 \end{bmatrix}$ and $K = [-0.3222 \quad -1.0011]$ Then, by solving W using LMI(5.29), we get the:

P_2 = $\begin{bmatrix} 3.6886 & -0.2459 \\ -0.2459 & 3.6208 \end{bmatrix}$, $A = \begin{bmatrix} 0.0628 & 0.9064 \\ 0.9258 & 0.0273 \end{bmatrix}$ and L = $\begin{bmatrix} 0.1920 & 0.8400 \\ 0.8400 & 0.2681 \end{bmatrix}$

Table 5.3: Eigenvalues of LMIs

Open loop system	Closed loop system
-0.8055,1.0055	-0.0611 ±0.2954i
-0.7514,1.0514	-0.0111 ±0.2804i
1.1000,-0.9000	-0.0611 ±0.3278i
1.1462,-0.8462	-0.0111 ±0.3144i
1.1000,-0.9000	-0.3656,0.2435
1.1462,-0.8462	-0.3295,0.3073
-1.0045,1.2045	-0.3966, 0.2744
-0.9511,1.2511	-0.3591,0.3370
-0.7220,1.1220	0.0389 ±0.2991i
-0.6624,1.1624	0.0889 ±0.3014i
1.2149,-0.8149	0.0389 ±0.3312i
1.2562,-0.7562	0.0889 ±0.3333i
1.2149,-0.8149	0.3398,-0.2620
1.2562, -0.7562	-0.2097,0.3875
-0.9180, 1.3180	0.3711,-0.2932
-0.8602, 1.3602	-0.2411,0.4190

Simulation results are shown in figures 5.10 ,5.11 ,5.12 and 5.13. Figure 5.10 shows the real and estimated state of $x_1(k)$ and $x_2(k)$. Figure 5.11 and figure 5.12 displays the estimation errors of the states $x_1(k)$ and $x_2(k)$ respectively . They show that the estimated states converge quickly to the real states with asymptotically stable error dynamics.The system eigenvalues with and without the proposed observer-based control are given in table 5.3. It is shown that the open loop system is unstable and by proposed methodology, all eigenvalues are inside the unit circle and the

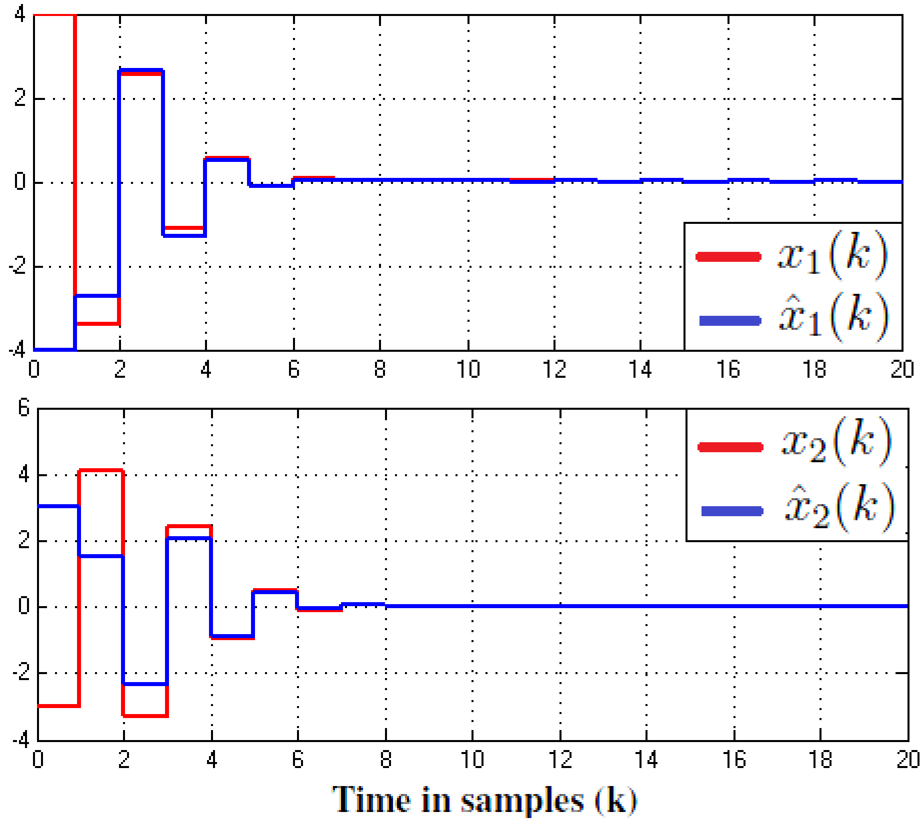

Figure 5.10: Real and estimated state response of the system

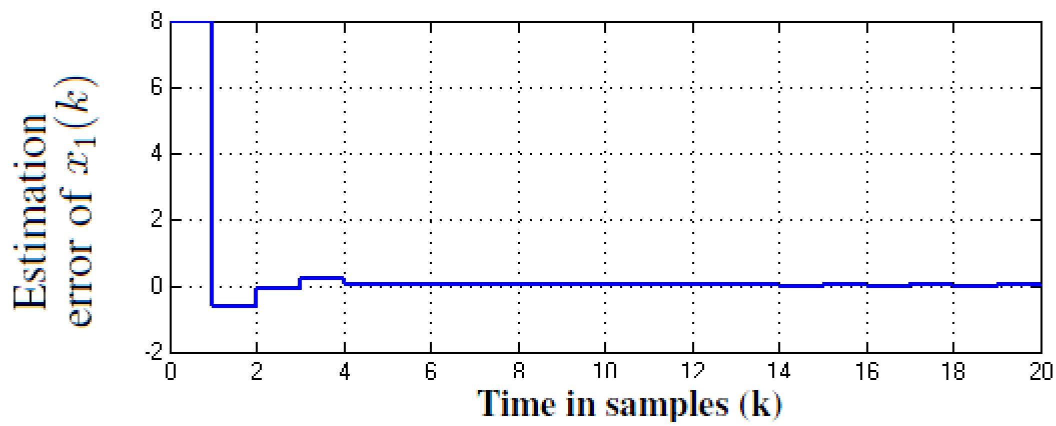

Figure 5.11: The time response error of the state $x_1(t)$

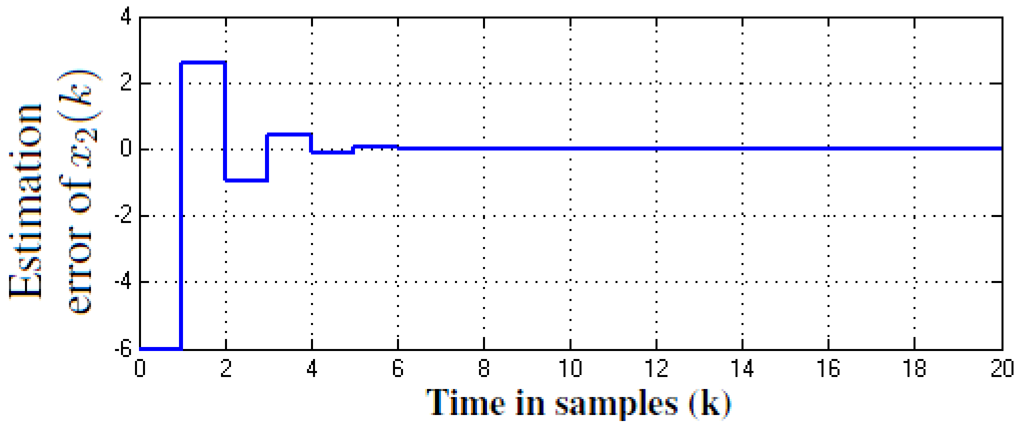

Figure 5.12: The time response error of the state $x_2(t)$

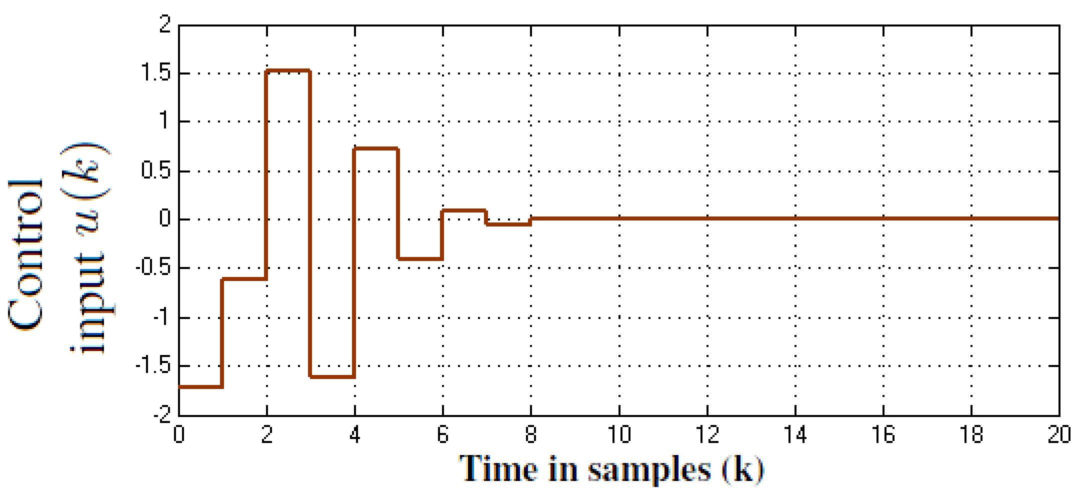

Figure 5.13: The time response of control input $u(t)$

feedback controller stabilize the system with closed loop eigenvalue. Finally, figure 5.13 shows the input control.

5.5 Discussion

In this chapter, the lyapunov stability theory and LMI approach are used to design of observer-based control of discrete time systems. The proposed methodology is to construct linear full order observer to guarantee the feedback controlled system is exponentially stabilizable. The results show it is fast and stabilize the system within few seconds. In [21], the separation principle methodology was used to design of observer-based control of discrete systems by D. Peaucelle and Y. Ebihara. The state-feedback gain is designed based on $\mathcal{H}_\infty$ control first. Then, the design of observer gain comes after that. However,the proposed methodology is to find

Table 5.4: The results of the parameters of the two methodologies

	Proposed Methodology	Peaucelle and Y. Ebihara Methodology
State-feedback gain(K)	$\begin{bmatrix}-0.8580 & -0.9441\end{bmatrix}$	$\begin{bmatrix}-1.0633 & -1.0324\end{bmatrix}$
Observer gain(L)	$\begin{bmatrix}2.5123 \\ 1.3826\end{bmatrix}$	$\begin{bmatrix}-2.3637 \\ -1.3565\end{bmatrix}$
System matrix A_0	$\begin{bmatrix}0.9452 & 0.9470 \\ 0.9900 & 0.0089\end{bmatrix}$	$\begin{bmatrix}0.9946 & 0.9807 \\ 0.9945 & -0.0195\end{bmatrix}$

the observer gain and state feedback gain in simultaneous. Both [43] and proposed methodology use convex optimization software (YALMIP toolbox). Figure 5.14 shows the state response of proposed methodology and [21] methodology.
Both of methodologies take around 17 second to reach convergence. Table 5.4 summarizes the results of the use of both methods.
Also,in[36], the separation principle methodology was used to design of observer-based control of discrete systems by W. Heemels, J. Daafouz and G. Millerioux.

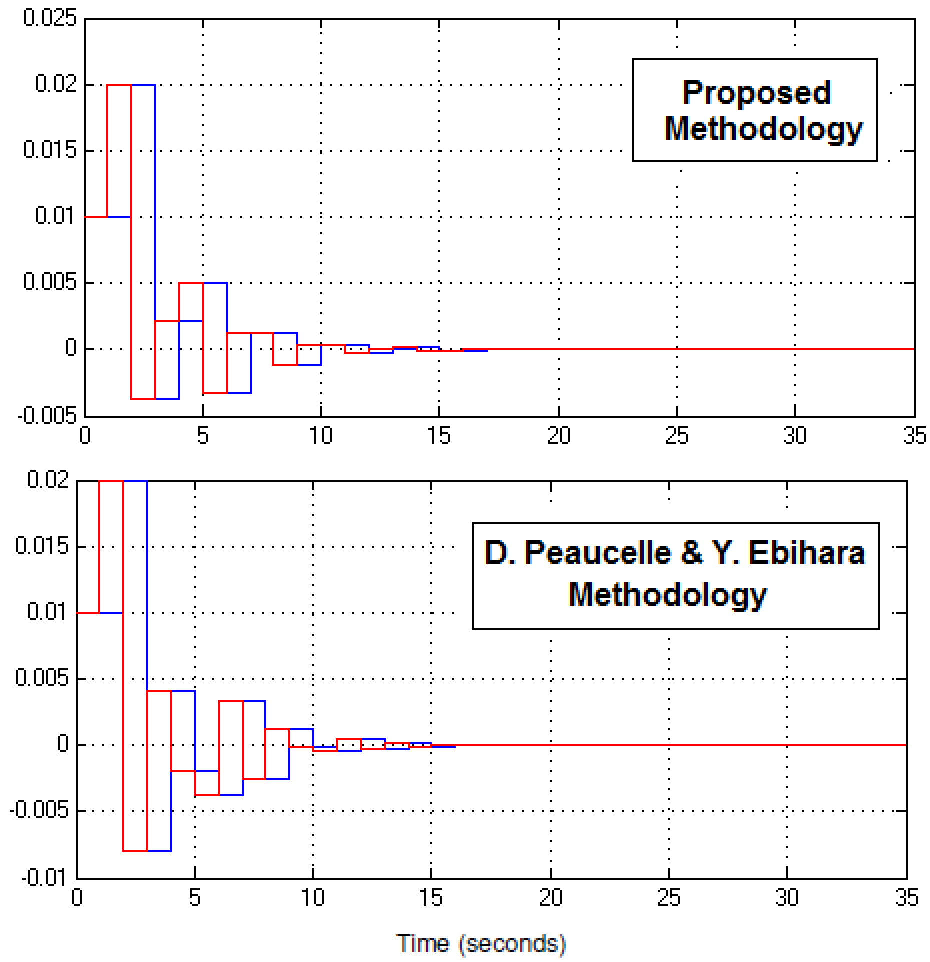

Figure 5.14: State response of proposed methodology and Peaucelle & Y. Ebihara methodology

The state-feedback gain and observer gain are designed based on input-to-state stability separately. However,the proposed methodology is to find the observer gain and state feedback gain in simultaneous. Table 5.5 summarizes the results of the use of both methods. Also, in this chapter, we suggest two methods to solve BMIs of observer-based control of discrete time systems. From figure 5.15, the first method. reaches convergence a little faster than second method(substitution method). Also, from the eigenvalue table 5.2 and table 5.3, all eigenvalues are inside the unit circle but the eigenvalue of first method larger than the eigenvalue of second method(substitution method).

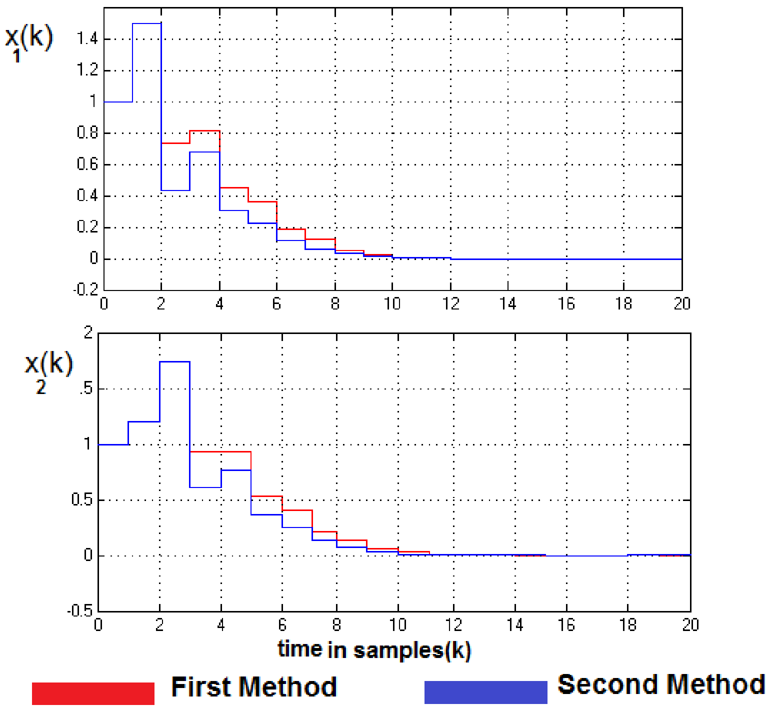

Figure 5.15: State response of the system example 5.2.2 by using both methods

Table 5.5: The results of the parameters of the two methodologies

	Proposed Methodology	W. Heemels, J. Daafouz & G. Millerioux Methodology
State-feedback gain controller	$K = [-0.061 \quad -0.241 \quad -0.314]$	$K_1 = [0.0327\ 0.1241\ 0.2387]$ $K_2 = [0.0005\ 0.0010\ 0.6148]$
Observer gain	$L = \begin{bmatrix} 0.0851 \\ 0.0497 \\ 0.4135 \end{bmatrix}$	$L_1 = \begin{bmatrix} 0.0835 \\ 0.0011 \\ 0.3870 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0.0835 \\ 0.0011 \\ 0.7094 \end{bmatrix}$

Chapter 6
Conclusion and future work

The main goal of this thesis is the development of a methodology to design observer-based control of continuous and discrete systems. The methodology based on lyapunov stability theory and convex optimization with LMI formulation. The LPV system is represented by polytopic model which is defined as convex hull of vertices. The design guarantee the feedback controlled system is exponentially stabilizable by the linear observer-based control. we found that the proposed methodology fast and stabilize the system much faster than published results. The LPV longitudinal model of helicopter has been developed and implemented in this thesis. It is used as example to implement the design of observer-based control of continuous time. To avoid getting equality constraint and infeasible solution sometimes, we suggest to solve the BMIs of observer-based control design by substitution method for both cases of continuous and discrete time systems. Numerical examples including the LPV longitudinal model of helicopter are given to illustrate the effectiveness of the proposed design results.

Although some results have been obtained, the work in this thesis can be further developed following some directions:

- Based on methodology proposed, the design of observer-based control will be extended to design-observer based control for LPV systems with state perturbations.
- More complicated and higher dimensional of LPV models can be considered as example to implment observer-based control.
- The convergence offered by the proposed methodology provable theoretically .
- There are different methods to transform BMI to LMI insted of using a change variables.

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APPENDIX A: Lyapunov Stability

Lyapunov stability is named after Aleksandr Lyapunov, a russian mathematician who published his book the general problem of stability of motion in 1892. Lyapunov was the first to consider the modifications necessary in nonlinear systems to the linear theory of stability based on linearizing near a point of equilibrium. His work, initially published in russian and then translated to french, received little attention for many years. Interest in it started suddenly during the Cold War period when the so-called ”Second Method of Lyapunov” was found to be applicable to the stability of aerospace guidance systems which typically contain strong nonlinearities not treatable by other methods. A large number of publications appeared then and since in the control and systems literature. More recently the concept of the Lyapunov exponent (related to Lyapunov’s First Method of discussing stability) has received wide interest in connection with chaos theory. Lyapunov stability methods have also been applied to finding equilibrium solutions in traffic assignment problems.

According to the lyapunov stability theory, the stability of dynamical systems can be determined in terms of certain scalar functions known as lyapunov functions(Halanay and Rasvan,1993). This can be done for both continuous time and discrete time domains. Consider a continuous time invariant linear system

$$\dot{x} = Ax(t), x(t_0) = x_0 \tag{6.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector and $A \in \mathbb{R}^{nxn}$ is the system matrix.

Theorem A.1 *the equilibrium point $x = 0$ of (A.1) a time invariant dynamical system is stable(in the sense of lyapunov) if there exists a continuously differentiable scalar function V(X) such that along the system trajectories the following is satisfied*

$$V(x) > 0, V(0) = 0 \tag{6.2}$$

$$\dot{V}(x) < 0 \tag{6.3}$$

if the condition (A.3) is a strict inequality then the equilibrium point $x = 0$ is asymptotically stable.
It is easy to show that for linear system(A.1) a lyapunov function can be chosen as a quadratic one, that is

$$V(x) = x^T P x, \;\; P = P^T > 0 \tag{6.4}$$

which with the use of (A.1) leads to

$$\dot{V}(x) = x^T(A^T P + PA)x \tag{6.5}$$

that is, the system is asymptotically stable if the following condition is satisfied

$$A^T P + PA < 0 \tag{6.6}$$

The theorem A.1 can be stated for the stability of discrete time systems. For linear discrete time system

$$x(k+1) = Ax(k), x(k_0) = x_0 \tag{6.7}$$

a lyapunov function has quadratic form which must satisfy(Kalman and Bertram,1960 and Ogata,1987)

$$V(k) = x^T(k)P(k) > 0 \tag{6.8}$$

$$\Delta V(k) = V(k+1) - V(k) < 0 \tag{6.9}$$

Since

$$V(k+1) - V(k) = x^T(k+1)Px(k+1) - x^T(k)Px(k) \tag{6.10}$$

$$= x^T(k)(A^T PA - P)x(k) < 0 \tag{6.11}$$

the system is asymptotically stable if the following condition is satisfied

$$A^T PA - P < 0 \tag{6.12}$$

APPENDIX B: Linear Matrix Inequality(LMI)

Linear matrix inequalities (LMIs) provide an incredibly powerful way to solve convex or quasi convex optimization problems. There is a long history of using LMI controllers in many different fields ranging from robotics, to electronics, to aerospace application. Their ability to provide robust control is constantly being proven. Many control problems require both performance and robustness objectives which can be solved using the LMI control because it allows for multi-objective optimization. The LMI is an expression of the form:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{6.13}$$

where the symmetric matrices $F_i = F_i^T \in R^{nxn}, i = 1, ...m$ is positive definite matrices and $x \in R^m$ is an unknown vector of scalar optimization variables.

Constraints expressed using LMI

The linear matrix inequality (B.1) defines a convex constraint on x. That is, the set $S := \{x|F(x) > 0\}$ is convex. Indeed, if $x_1, x_2 \in S$ and $\alpha \in (0, 1)$ then

$$F(\alpha x_1 + (1 - \alpha)x_2) = \alpha F(x_1) + (1 - \alpha)F(x_2) > 0 \tag{6.14}$$

where in the equality we used that F is affine and the inequality follows from the fact that $\alpha > 0$ and $(1 - \alpha) \geq 0$.

Although the convex constraint $F(x) > 0$ on x may seem rather special, it turns out that many convex sets can be represented in this way. In this subsection we discuss some seemingly trivial properties of linear matrix inequalities which turn out to be of eminent help in the reduction of multiple constraints on an unknown variable to an equivalent constraint involving a single linear matrix inequality.

Definition B.1 (System of LMIs). A system of linear matrix inequalities is a finite set of linear matrix inequalities

$$F_1(x) > 0, F_2(x) > 0, ..., F_k(x) > 0 \tag{6.15}$$

It is a simple but essential property that every system of LMIs can be rewritten as one single LMI. Specifically, $F_1(x) > 0, F_2(x) > 0, ..., F_k(x) > 0$ if and only if

$$F(x) = \begin{bmatrix} F_1(x) & 0 & .... & 0 \\ 0 & F_2(x) & .... & 0 \\ : & : & .... & : \\ 0 & 0 & .... & F_k(x) \end{bmatrix} > 0 \tag{6.16}$$

The last inequality indeed makes sense as $F(x)$ is symmetric for any x. Furthermore, since the set of eigenvalues of $F(x)$ is simply the union of the eigenvalues of $F_1(x), ..., F_k(x)$, any x that satisfies $F(x) > 0$ also satisfies the system of LMIs (B.3) and vice versa.

Properties of Positive Definite Matrices

- Addition of positive matrices: $A > 0$ and $B > 0 \Rightarrow A + B > 0$
- Block diagonal matrices: $A > 0$ and $B > 0 \Leftrightarrow \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} > 0$
- Invertibility: $A > 0 \Rightarrow A$ nonsingular.
- If $P = P^T$ then P^{-1} always exists, and moreover $P^{-1} = (P^{-1})^T > 0$

Definition B.2. Feasibility Problem

The feasibility problems concern finding elements $x \in X$such that $F(x) < 0$. The

LMI $F(x) < 0$ is called feasible if such elements x exist, otherwise it is said to be infeasible.

Schur Complement

Schur complement is a very useful tool to modify certain linear matrix inequality constraints to a different form that is more suitable for the particular algorithm.

Lemma 1 (Schur Complement). suppose R and S are symmetric, i.e. $R = R^T$ and $S = S^T$. Then, the following conditions are equivalent:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0 \Leftrightarrow \begin{cases} R < 0, Q - SR^{-1}S^T < 0 \\ \\ Q < 0, R - S^T R^{-1} S < 0 \end{cases} \tag{6.17}$$

The schur complement result can be generalized to nonstrict inequalities. Using schur complements we can infer that if a matrix is positive definite then an arbitrary diagonal square sub block is also positive definite. For instance, if any diagonal element of a matrix P is negative or zero the matrix P is not positive definite.

For example, consider an algebraic riccati inequality:

$$A^T P + PA + PBR^{-1}B^T + Q < 0 \tag{6.18}$$

where A, B, $Q = Q^T$ and $R = R^T > 0$ are given matrices and $P = P^T$ is the variable.Then, the inequality (B.5) is equivalent to the following LMI:

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} < 0 \tag{6.19}$$

Bilinear matrix inequality (BMI)

A bilinear matrix inequality (BMI) is an inequality of the form

$$F(x, y) \triangleq F_0 + \sum_{i=1}^{m_x} x_i F_i + \sum_{i=1}^{m_y} y_i G_i + \sum_{i=1}^{m_y} \sum_{i=1}^{m_x} y_i x_i H_{ij} \geq 0 \tag{6.20}$$

The variables are $x \in \mathbb{R}^{m_x}$ and $y \in \mathbb{R}^{m_y}$. The matrices $F_i = F_i^T \in \mathbb{R}^{nxn}$, $i = 1, ..., m_x$, $G_i = G_i^T \in \mathbb{R}^{nxn}$ and $j = 1, ..., m_y$ and $H_{ij} = H_{ij}^T \in \mathbb{R}^{nxn}$, $i = 1, ..., m_x, j = 1, ..., m_y$ are are given data, and the inequality sign '$\geq$' denotes positive semi definiteness. For y fixed y the BMI (B.7) reduces to a linear matrix inequality (LMI) in the variable x for fixed x it reduces to an LMI in the variable y. BMIs were introduced by Safonov, Goh, Mesbahi, and others as a unified description of a wide variety of control problems. We will consider the following BMI optimization problem:

$$\begin{aligned} minimize f(x, y) = c^T x + d^T y \\ subject\, to\, F(x, y) \geq 0 \\ A(y) = A_0 + \sum_{i=1}^{m_y} A_j \geq 0 \\ -l_i \leq x_i \leq u_i, i = 1, ..., m_x \end{aligned} \tag{6.21}$$

where the variables x and y . $A_i = A_i^T \in \mathbb{R}^{pxp}$, $i = 1, ..., m_y$. The set $\{y | A(y) \geq 0\} \equiv$ is assumed to be bounded.

APPENDIX C: MATLAB Codes

```
clear
%Example: 4.4.1
%System: 4th Order-continuous time- 3 uncertain parameters
%Solved: by first method
A1=[-3 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -5 ];
A2=[-3 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -6 ];
A3=[-3 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -5 ];
A4=[-3 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -6 ];
A5=[-5 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -5 ];
A6=[-5 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -6 ];
A7=[-5 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -5 ];
A8=[-5 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -6 ];
B=[ 1;0;1 ;0 ];
C = [1 0 0 0];
P1 = sdpvar(4,4,'symmetric');
P2 = sdpvar(4,4,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(1,4);
X = sdpvar(4,4);
Y = sdpvar(4,1);
F = lmi(P1>0) ;
F =F + lmi(P2>0) ;
```

```
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([A1'*P1+G'*B'+P1*A1+B*G  A1'*P2-X'-B*G ; -G'*B'+P2*A1-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A2'*P1+G'*B'+P1*A2+B*G  A2'*P2-X'-B*G ; -G'*B'+P2*A2-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A3'*P1+G'*B'+P1*A3+B*G  A3'*P2-X'-B*G ; -G'*B'+P2*A3-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A4'*P1+G'*B'+P1*A4+B*G  A4'*P2-X'-B*G ; -G'*B'+P2*A4-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A5'*P1+G'*B'+P1*A5+B*G  A5'*P2-X'-B*G ; -G'*B'+P2*A5-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A6'*P1+G'*B'+P1*A6+B*G  A6'*P2-X'-B*G ; -G'*B'+P2*A6-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A7'*P1+G'*B'+P1*A7+B*G  A7'*P2-X'-B*G ; -G'*B'+P2*A7-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A8'*P1+G'*B'+P1*A8+B*G  A8'*P2-X'-B*G ; -G'*B'+P2*A8-X
X'-C'*Y'+X-Y*C] <0)
solvesdp(F)
A0=inv(double(P2))*double(X)
K=inv(double(Z))*double(G)
L=inv(double(P2))*double(Y)
```

```
clear
%Example: 4.4.1
%System: 4th Order-continuous time- 3 uncertain parameters
%Solved : by second method(substitution method)
A1=[-3 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -5 ];
A2=[-3 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -6 ];
A3=[-3 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -5 ];
A4=[-3 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -6 ];
A5=[-5 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -5 ];
A6=[-5 0 1 0;0 -2 0 1; 0.7 0 1 0;1 0 2 -6 ];
A7=[-5 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -5 ];
A8=[-5 0 1 0;0 -3 0 1; 0.7 0 1 0;1 0 2 -6 ];
B=[ 1;0;1 ;0 ];
C = [1 0 0 0];
X1 = sdpvar(4,4,'symmetric');
Y1 = sdpvar(1,4);
F = lmi(X1>0) ;
F =F + lmi([X1*A1'+A1*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A2'+A2*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A3'+A3*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A4'+A4*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A5'+A5*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A6'+A6*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A7'+A7*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A8'+A8*X1+B*Y1+Y1'*B']<0);
solvesdp(F)
P1=inv(double(X1))
K=double(Y1)*double(P1)
P2 = sdpvar(4,4,'symmetric');
X2 = sdpvar(4,4);
```

```
Y2 = sdpvar(4,1);
F = lmi(P2>0) ;
F =F + lmi([A1'*P1+K'*B'*P1+P1*A1+P1*B*K A1'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A1-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A2'*P1+K'*B'*P1+P1*A2+P1*B*K A2'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A2-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A3'*P1+K'*B'*P1+P1*A3+P1*B*K A3'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A3-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A4'*P1+K'*B'*P1+P1*A4+P1*B*K A4'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A4-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A5'*P1+K'*B'*P1+P1*A5+P1*B*K A5'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A5-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A6'*P1+K'*B'*P1+P1*A6+P1*B*K A6'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A6-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A7'*P1+K'*B'*P1+P1*A7+P1*B*K A7'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A7-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A8'*P1+K'*B'*P1+P1*A8+P1*B*K A8'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A8-X2 X2'-C'*Y2'+X2-Y2*C]<0)
solvesdp(F)
A0=inv(double(P2))*double(X2)
L=inv(double(P2))*double(Y2)
```

```
clear
%Example :4.4.2
% System: longitudinal model of helicopter-continuous time
% Solved: by first method
thetae=[-15*3.14/180; 15*3.14/180];% uncertain parameter
phie=[-15*3.14/180; 15*3.14/180];% uncertain parameter
we=[-5.5;5.5];% uncertain parameter
g=9.81;
Tmrm=[0.8*g;1.2*g];% uncertain parameter
Xu=0.0321;Mu=0;
invte=8.36;
Ma=204;
DD=-0.0076;
A1=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A2=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A3=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A4=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A5=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A6=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A7=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A8=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A9=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0 -1
```

```
-invte ];
A10=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A11=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A12=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A13=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A14=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A15=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A16=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
B= [1 0; 0 1;0 2;0.7 4];;
C=[1 0 0 0;0 1 0 0;0 0 1 0 ;0 0 0 1 ];
P1 = sdpvar(4,4,'symmetric');
P2 = sdpvar(4,4,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(2,4);
X = sdpvar(4,4);
Y = sdpvar(4,4);
F = lmi(P1>0) ;
F =F + lmi(P2>0) ;
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([A1'*P1+G'*B'+P1*A1+B*G A1'*P2-X'-B*G ; -G'*B'+P2*A1-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A2'*P1+G'*B'+P1*A2+B*G A2'*P2-X'-B*G ; -G'*B'+P2*A2-X
X'-C'*Y'+X-Y*C] <0)
```

```
F =F + lmi([A3'*P1+G'*B'+P1*A3+B*G  A3'*P2-X'-B*G ; -G'*B'+P2*A3-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A4'*P1+G'*B'+P1*A4+B*G  A4'*P2-X'-B*G ; -G'*B'+P2*A4-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A5'*P1+G'*B'+P1*A5+B*G  A5'*P2-X'-B*G ; -G'*B'+P2*A5-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A6'*P1+G'*B'+P1*A6+B*G  A6'*P2-X'-B*G ; -G'*B'+P2*A6-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A7'*P1+G'*B'+P1*A7+B*G  A7'*P2-X'-B*G ; -G'*B'+P2*A7-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A8'*P1+G'*B'+P1*A8+B*G  A8'*P2-X'-B*G ; -G'*B'+P2*A8-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A9'*P1+G'*B'+P1*A9+B*G  A9'*P2-X'-B*G ; -G'*B'+P2*A9-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A10'*P1+G'*B'+P1*A10+B*G A10'*P2-X'-B*G ; -G'*B'+P2*A10-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A11'*P1+G'*B'+P1*A11+B*G A11'*P2-X'-B*G ; -G'*B'+P2*A11-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A12'*P1+G'*B'+P1*A12+B*G A12'*P2-X'-B*G ; -G'*B'+P2*A12-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A13'*P1+G'*B'+P1*A13+B*G A13'*P2-X'-B*G ; -G'*B'+P2*A13-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A14'*P1+G'*B'+P1*A14+B*G A14'*P2-X'-B*G ; -G'*B'+P2*A14-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A15'*P1+G'*B'+P1*A15+B*G A15'*P2-X'-B*G ; -G'*B'+P2*A15-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A16'*P1+G'*B'+P1*A16+B*G A16'*P2-X'-B*G ; -G'*B'+P2*A16-
X X'-C'*Y'+X-Y*C] <0)
solvesdp(F)
A0=inv(double(P2))*double(X)
```

```
K=inv(double(Z))*double(G)
L=inv(double(P2))*double(Y)
```

```
clear
%Example :4.4.2
% System: longitudinal model of helicopter-continuous time
% Solved: by second method(substitution method)
thetae=[-15*3.14/180; 15*3.14/180];% uncertain parameter
phie=[-15*3.14/180; 15*3.14/180];% uncertain parameter
we=[-5.5;5.5];% uncertain parameter
g=9.81;
Tmrm=[0.8*g;1.2*g];% uncertain parameter
Xu=0.0321;Mu=0; invte=8.36; Ma=204; DD=-0.0076;
A1=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A2=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A3=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A4=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(1);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A5=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A6=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A7=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A8=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(2);0 0 cos(phie(1)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A9=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0 -1
-invte ];
A10=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
```

```
A11=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A12=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(1);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A13=[Xu -g*cos(thetae(1)) -we(1) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A14=[Xu -g*cos(thetae(2)) -we(1) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A15=[Xu -g*cos(thetae(1)) -we(2) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
A16=[Xu -g*cos(thetae(2)) -we(2) -Tmrm(2);0 0 cos(phie(2)) 0;Mu 0 0 Ma;DD 0
-1 -invte ];
B= [1 0; 0 1;0 2;0.7 4];;
C=[1 0 0 0;0 1 0 0;0 0 1 0 ;0 0 0 1 ];
X1 = sdpvar(4,4,'symmetric');
Y1 = sdpvar(2,4);
F = lmi(X1>0) ;
F =F + lmi([X1*A1'+A1*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A2'+A2*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A3'+A3*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A4'+A4*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A5'+A5*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A6'+A6*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A7'+A7*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A8'+A8*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A9'+A9*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A10'+A10*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A11'+A11*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A12'+A12*X1+B*Y1+Y1'*B']<0);
F =F + lmi([X1*A13'+A13*X1+B*Y1+Y1'*B']<0);
```

```
F =F + lmi([X1*A14’+A14*X1+B*Y1+Y1’*B’]<0);
F =F + lmi([X1*A15’+A15*X1+B*Y1+Y1’*B’]<0);
F =F + lmi([X1*A16’+A16*X1+B*Y1+Y1’*B’]<0);
solvesdp(F)
P1=inv(double(X1))
K=double(Y1)*double(P1)
P2 = sdpvar(4,4,’symmetric’);
X2 = sdpvar(4,4);
Y2 = sdpvar(4,4);
F = lmi(P2>0) ;
F =F + lmi([A1’*P1+K’*B’*P1+P1*A1+P1*B*K A1’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A1-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A2’*P1+K’*B’*P1+P1*A2+P1*B*K A2’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A2-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A3’*P1+K’*B’*P1+P1*A3+P1*B*K A3’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A3-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A4’*P1+K’*B’*P1+P1*A4+P1*B*K A4’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A4-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A5’*P1+K’*B’*P1+P1*A5+P1*B*K A5’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A5-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A6’*P1+K’*B’*P1+P1*A6+P1*B*K A6’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A6-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A7’*P1+K’*B’*P1+P1*A7+P1*B*K A7’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A7-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A8’*P1+K’*B’*P1+P1*A8+P1*B*K A8’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A8-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A9’*P1+K’*B’*P1+P1*A9+P1*B*K A9’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A9-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
F =F + lmi([A10’*P1+K’*B’*P1+P1*A10+P1*B*K A10’*P2-X2’-P1*B*K ; -K’*B’*P1+P2*A10-
X2 X2’-C’*Y2’+X2-Y2*C]<0)
```

```
F =F + lmi([A11'*P1+K'*B'*P1+P1*A11+P1*B*K A11'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A11-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A12'*P1+K'*B'*P1+P1*A12+P1*B*K A12'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A12-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A13'*P1+K'*B'*P1+P1*A13+P1*B*K A13'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A13-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A14'*P1+K'*B'*P1+P1*A14+P1*B*K A14'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A14-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A15'*P1+K'*B'*P1+P1*A15+P1*B*K A15'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A15-X2 X2'-C'*Y2'+X2-Y2*C]<0)
F =F + lmi([A16'*P1+K'*B'*P1+P1*A16+P1*B*K A16'*P2-X2'-P1*B*K ; -K'*B'*P1+P2*A16-X2 X2'-C'*Y2'+X2-Y2*C]<0)
solvesdp(F)
A0=inv(double(P2))*double(X2)
L=inv(double(P2))*double(Y2)
```

```
clear
%Example :4.4.3
% System: 2nd order-continuous time-All elements of system matrix are uncertain
parameters
% Solved: by first method
A1=[ -6 1;10 0];
A2=[ -6 1; 10 0.7]; A3=[ -6 1;20 0];
A4=[ -6 1; 20 0.7]; A5=[ -6 2.4;10 0];
A6=[ -6 2.4; 10 0.7]; A7=[ -6 2.4;20 0];
A8=[ -6 2.4; 20 0.7]; A9=[ -3 1;10 0];
A10=[ -3 1; 10 0.7]; A11=[ -3 1;20 0];
A12=[ -3 1; 20 0.7]; A13=[ -3 2.4;10 0];
A14=[ -3 2.4; 10 0.7]; A15=[ -3 2.4;20 0];
A16=[ -3 2.4; 20 0.7];
B=[ 0; 1 ];
C=[ 1 1 ];
P1 = sdpvar(2,2,'symmetric');
P2 = sdpvar(2,2,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(1,2);
X = sdpvar(2,2);
Y = sdpvar(2,1);
F = lmi(P1 >0) ;
F =F + lmi(P2>0) ;
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([A1'*P1+G'*B'+P1*A1+B*G  A1'*P2-X'-B*G ; -G'*B'+P2*A1-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A2'*P1+G'*B'+P1*A2+B*G  A2'*P2-X'-B*G ; -G'*B'+P2*A2-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A3'*P1+G'*B'+P1*A3+B*G  A3'*P2-X'-B*G ; -G'*B'+P2*A3-X
```

```
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A4'*P1+G'*B'+P1*A4+B*G  A4'*P2-X'-B*G ; -G'*B'+P2*A4-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A5'*P1+G'*B'+P1*A5+B*G  A5'*P2-X'-B*G ; -G'*B'+P2*A5-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A6'*P1+G'*B'+P1*A6+B*G  A6'*P2-X'-B*G ; -G'*B'+P2*A6-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A7'*P1+G'*B'+P1*A7+B*G  A7'*P2-X'-B*G ; -G'*B'+P2*A7-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A8'*P1+G'*B'+P1*A8+B*G  A8'*P2-X'-B*G ; -G'*B'+P2*A8-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A9'*P1+G'*B'+P1*A9+B*G  A9'*P2-X'-B*G ; -G'*B'+P2*A9-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A10'*P1+G'*B'+P1*A10+B*G A10'*P2-X'-B*G ; -G'*B'+P2*A10-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A11'*P1+G'*B'+P1*A11+B*G A11'*P2-X'-B*G ; -G'*B'+P2*A11-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A12'*P1+G'*B'+P1*A12+B*G A12'*P2-X'-B*G ; -G'*B'+P2*A12-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A13'*P1+G'*B'+P1*A13+B*G A13'*P2-X'-B*G ; -G'*B'+P2*A13-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A14'*P1+G'*B'+P1*A14+B*G A14'*P2-X'-B*G ; -G'*B'+P2*A14-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A15'*P1+G'*B'+P1*A15+B*G A15'*P2-X'-B*G ; -G'*B'+P2*A15-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A16'*P1+G'*B'+P1*A16+B*G A16'*P2-X'-B*G ; -G'*B'+P2*A16-
X X'-C'*Y'+X-Y*C] <0)
solvesdp(F)
A0=inv(double(P2))*double(X)
K=inv(double(Z))*double(G)
```

```
L=inv(double(P2))*double(Y)
clear
%Example :4.3.3
% System: 2nd order-continuous time-All elements of system matrix are uncertain
parameters
% Solved: by second method(substitution method)
A1=[ -6 1;10 0];
A2=[ -6 1; 10 0.7]; A3=[ -6 1;20 0];
A4=[ -6 1; 20 0.7]; A5=[ -6 2.4;10 0];
A6=[ -6 2.4; 10 0.7]; A7=[ -6 2.4;20 0];
A8=[ -6 2.4; 20 0.7]; A9=[ -3 1;10 0];
A10=[ -3 1; 10 0.7]; A11=[ -3 1;20 0];
A12=[ -3 1; 20 0.7]; A13=[ -3 2.4;10 0];
A14=[ -3 2.4; 10 0.7]; A15=[ -3 2.4;20 0];
A16=[ -3 2.4; 20 0.7];
B=[ 0; 1 ]; C=[ 1 1 ];
P1 = sdpvar(2,2,'symmetric');
P2 = sdpvar(2,2,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(1,2);
X = sdpvar(2,2);
Y = sdpvar(2,1);
F = lmi(P1>0) ;
F =F + lmi(P2>0) ;
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([A1'*P1+G'*B'+P1*A1+B*G A1'*P2-X'-B*G ; -G'*B'+P2*A1-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A2'*P1+G'*B'+P1*A2+B*G A2'*P2-X'-B*G ; -G'*B'+P2*A2-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A3'*P1+G'*B'+P1*A3+B*G A3'*P2-X'-B*G ; -G'*B'+P2*A3-X
```

```
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A4'*P1+G'*B'+P1*A4+B*G  A4'*P2-X'-B*G ; -G'*B'+P2*A4-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A5'*P1+G'*B'+P1*A5+B*G  A5'*P2-X'-B*G ; -G'*B'+P2*A5-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A6'*P1+G'*B'+P1*A6+B*G  A6'*P2-X'-B*G ; -G'*B'+P2*A6-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A7'*P1+G'*B'+P1*A7+B*G  A7'*P2-X'-B*G ; -G'*B'+P2*A7-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A8'*P1+G'*B'+P1*A8+B*G  A8'*P2-X'-B*G ; -G'*B'+P2*A8-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A9'*P1+G'*B'+P1*A9+B*G  A9'*P2-X'-B*G ; -G'*B'+P2*A9-X
X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A10'*P1+G'*B'+P1*A10+B*G A10'*P2-X'-B*G ; -G'*B'+P2*A10-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A11'*P1+G'*B'+P1*A11+B*G A11'*P2-X'-B*G ; -G'*B'+P2*A11-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A12'*P1+G'*B'+P1*A12+B*G A12'*P2-X'-B*G ; -G'*B'+P2*A12-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A13'*P1+G'*B'+P1*A13+B*G A13'*P2-X'-B*G ; -G'*B'+P2*A13-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A14'*P1+G'*B'+P1*A14+B*G A14'*P2-X'-B*G ; -G'*B'+P2*A14-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A15'*P1+G'*B'+P1*A15+B*G A15'*P2-X'-B*G ; -G'*B'+P2*A15-
X X'-C'*Y'+X-Y*C] <0)
F =F + lmi([A16'*P1+G'*B'+P1*A16+B*G A16'*P2-X'-B*G ; -G'*B'+P2*A16-
X X'-C'*Y'+X-Y*C] <0)
solvesdp(F)
A0=inv(double(P2))*double(X)
K=inv(double(Z))*double(G)
```

```
L=inv(double(P2))*double(Y)

    %Example :5.4.1
%System: 4th Order-discrete time- 3 uncertain parameters
% Solved: by first method
clear
A1=[0.2 0.2 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 0.1 0 -0.7];
A2=[0.2 0.2 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 1.5 0 -0.7];
A3=[0.2 0.7 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 0.1 0 -0.7];
A4=[0.2 0.7 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 1.5 0 -0.7];
A5=[0.8 0.2 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 0.1 0 -0.7];
A6=[0.8 0.2 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 1.5 0 -0.7];
A7=[0.8 0.7 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 0.1 0 -0.7];
A8=[0.8 0.7 -2 0; 0.5 0 1 0; -0.8 0 -0.3 0;1 0.5 0 -0.7];
B =[1.1 0;0 1;0 0;0 0] ;
C =[1 0 0 0;0 0 1 0];
P1 = sdpvar(4,4,'symmetric');
P2 = sdpvar(4,4,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(2,4);
X = sdpvar(4,4);
Y = sdpvar(4,2);
F = lmi(P1>0) ;
F =F + lmi(P2>0) ;
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([P1 zeros(4,4) A1'*P1+G'*B' A1'*P2-X';zeros(4,4) P2 -G'*B' X'-
C'*Y';P1*A1+B*G -B*G P1 zeros(4,4) ; P2*A1-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A2'*P1+G'*B' A2'*P2-X';zeros(4,4) P2 -G'*B' X'-
C'*Y';P1*A2+B*G -B*G P1 zeros(4,4) ; P2*A2-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A3'*P1+G'*B' A3'*P2-X';zeros(4,4) P2 -G'*B' X'-
```

```
C’*Y’;P1*A3+B*G -B*G P1 zeros(4,4) ; P2*A3-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A4’*P1+G’*B’ A4’*P2-X’;zeros(4,4) P2 -G’*B’ X’-
C’*Y’;P1*A4+B*G -B*G P1 zeros(4,4) ; P2*A4-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A5’*P1+G’*B’ A5’*P2-X’;zeros(4,4) P2 -G’*B’ X’-
C’*Y’;P1*A5+B*G -B*G P1 zeros(4,4) ; P2*A5-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A6’*P1+G’*B’ A6’*P2-X’;zeros(4,4) P2 -G’*B’ X’-
C’*Y’;P1*A6+B*G -B*G P1 zeros(4,4) ; P2*A6-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A7’*P1+G’*B’ A7’*P2-X’;zeros(4,4) P2 -G’*B’ X’-
C’*Y’;P1*A7+B*G -B*G P1 zeros(4,4) ; P2*A7-X X-Y*C zeros(4,4) P2] > 0)
F =F + lmi([P1 zeros(4,4) A8’*P1+G’*B’ A8’*P2-X’;zeros(4,4) P2 -G’*B’ X’-
C’*Y’;P1*A8+B*G -B*G P1 zeros(4,4) ; P2*A8-X X-Y*C zeros(4,4) P2] > 0)
solvesdp(F)
A0=inv(double(P2))*double(X)
K=inv(double(Z))*double(G)
L=inv(double(P2))*double(Y)
```

```
%Example :5.4.2
% System: 2nd order-discrete time-All elements of system matrix are uncertain parameters
% Solved: by first method
clear a1=0.2;a2=0.4; b1=0.9;b2=1.1;
c1=0.9 ;c2=1.1; d1=0;d2=0.1;
A1=[a1 b1;c1 d1]; A2=[a1 b1;c1 d2];
A3=[a1 b1;c2 d1]; A4=[a1 b1;c2 d2];
A5=[a1 b2;c1 d1]; A6=[a1 b2;c1 d2];
A7=[a1 b2;c2 d1]; A8=[a1 b2;c2 d2];
A9=[a2 b1;c1 d1]; A10=[a2 b1;c1 d2];
A11=[a2 b1;c2 d1]; A12=[a2 b1;c2 d2];
A13=[a2 b2;c1 d1]; A14=[a2 b2;c1 d2];
A15=[a2 b2;c2 d1]; A16=[a2 b2;c2 d2];
B =[ 1;0] ; C =[ 1 0;0 1];
P1 = sdpvar(2,2,'symmetric');
P2 = sdpvar(2,2,'symmetric');
Z = sdpvar(1,1);
G = sdpvar(1,2);
X = sdpvar(2,2); Y = sdpvar(2,2);
F = lmi(P1>0) ;
F =F + lmi(P2>0) ;
F =F + lmi(P1*B==B*Z ) ;
F =F + lmi([P1 zeros(2,2) A1'*P1+G'*B' A1'*P2-X';zeros(2,2) P2 -G'*B' X'-C'*Y';P1*A1+B*G -B*G P1 zeros(2,2) ; P2*A1-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A2'*P1+G'*B' A2'*P2-X';zeros(2,2) P2 -G'*B' X'-C'*Y';P1*A2+B*G -B*G P1 zeros(2,2) ; P2*A2-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A3'*P1+G'*B' A3'*P2-X';zeros(2,2) P2 -G'*B' X'-C'*Y';P1*A3+B*G -B*G P1 zeros(2,2) ; P2*A3-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A4'*P1+G'*B' A4'*P2-X';zeros(2,2) P2 -G'*B' X'-
```

```
C'*Y';P1*A4+B*G -B*G P1 zeros(2,2) ; P2*A4-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A5'*P1+G'*B' A5'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A5+B*G -B*G P1 zeros(2,2) ; P2*A5-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A6'*P1+G'*B' A6'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A6+B*G -B*G P1 zeros(2,2) ; P2*A6-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A7'*P1+G'*B' A7'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A7+B*G -B*G P1 zeros(2,2) ; P2*A7-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A8'*P1+G'*B' A8'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A8+B*G -B*G P1 zeros(2,2) ; P2*A8-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A9'*P1+G'*B' A9'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A9+B*G -B*G P1 zeros(2,2) ; P2*A9-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A10'*P1+G'*B' A10'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A10+B*G -B*G P1 zeros(2,2) ; P2*A10-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A11'*P1+G'*B' A11'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A11+B*G -B*G P1 zeros(2,2) ; P2*A11-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A12'*P1+G'*B' A12'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A12+B*G -B*G P1 zeros(2,2) ; P2*A12-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A13'*P1+G'*B' A13'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A13+B*G -B*G P1 zeros(2,2) ; P2*A13-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A14'*P1+G'*B' A14'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A14+B*G -B*G P1 zeros(2,2) ; P2*A14-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A15'*P1+G'*B' A15'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A15+B*G -B*G P1 zeros(2,2) ; P2*A15-X X-Y*C zeros(2,2) P2] > 0)
F =F + lmi([P1 zeros(2,2) A16'*P1+G'*B' A16'*P2-X';zeros(2,2) P2 -G'*B' X'-
C'*Y';P1*A16+B*G -B*G P1 zeros(2,2) ; P2*A16-X X-Y*C zeros(2,2) P2] > 0)
solvesdp(F)
A0=inv(double(P2))*double(X)
K=inv(double(Z))*double(G)
L=inv(double(P2))*double(Y)
```

```
%Example :5.4.2
% System: 2nd order-discrete time-All elements of system matrix are uncertain parameters
% Solved: by second method(substitution method)
clear a1=0.2;a2=0.4; b1=0.9;b2=1.1;
c1=0.9 ;c2=1.1; d1=0;d2=0.1;
A1=[a1 b1;c1 d1]; A2=[a1 b1;c1 d2];
A3=[a1 b1;c2 d1]; A4=[a1 b1;c2 d2];
A5=[a1 b2;c1 d1]; A6=[a1 b2;c1 d2];
A7=[a1 b2;c2 d1]; A8=[a1 b2;c2 d2];
A9=[a2 b1;c1 d1]; A10=[a2 b1;c1 d2];
A11=[a2 b1;c2 d1]; A12=[a2 b1;c2 d2];
A13=[a2 b2;c1 d1]; A14=[a2 b2;c1 d2];
A15=[a2 b2;c2 d1]; A16=[a2 b2;c2 d2];
B =[ 1;0] ; C =[ 1 0;0 1];
S = sdpvar(2,2,'symmetric');
Q = sdpvar(1,2);
F = lmi(S>0) ;
F =F + lmi([-S S*A1'+Q'*B';A1*S+B*Q -S]<0);
F =F + lmi([-S S*A2'+Q'*B';A2*S+B*Q -S]<0);
F =F + lmi([-S S*A3'+Q'*B';A3*S+B*Q -S]<0);
F =F + lmi([-S S*A4'+Q'*B';A4*S+B*Q -S]<0);
F =F + lmi([-S S*A5'+Q'*B';A5*S+B*Q -S]<0);
F =F + lmi([-S S*A6'+Q'*B';A6*S+B*Q -S]<0);
F =F + lmi([-S S*A7'+Q'*B';A7*S+B*Q -S]<0);
F =F + lmi([-S S*A8'+Q'*B';A8*S+B*Q -S]<0);
F =F + lmi([-S S*A9'+Q'*B';A9*S+B*Q -S]<0);
F =F + lmi([-S S*A10'+Q'*B';A10*S+B*Q -S]<0);
F =F + lmi([-S S*A11'+Q'*B';A11*S+B*Q -S]<0);
F =F + lmi([-S S*A12'+Q'*B';A12*S+B*Q -S]<0);
```

```
F =F + lmi([-S S*A13'+Q'*B';A13*S+B*Q -S]<0);
F =F + lmi([-S S*A14'+Q'*B';A14*S+B*Q -S]<0);
F =F + lmi([-S S*A15'+Q'*B';A15*S+B*Q -S]<0);
F =F + lmi([-S S*A16'+Q'*B';A16*S+B*Q -S]<0);
solvesdp(F)
P1=inv(double(S))
K=double(Q)*double(P1)
P2 = sdpvar(2,2,'symmetric');
W = sdpvar(2,2);
F2 = lmi(P2>0) ;
F2 =F2 + lmi([A1'*P1*A1+A1'*P1*B*K+K'*B'*P1*A1+K'*B'*P1*B*K+A1'*P2*A1-
A1'*W-W'*A1-P1 ]<0);
F2 =F2 + lmi([A2'*P1*A2+A2'*P1*B*K+K'*B'*P1*A2+K'*B'*P1*B*K+A2'*P2*A2-
A2'*W-W'*A2-P1 ]<0);
F2 =F2 + lmi([A3'*P1*A3+A3'*P1*B*K+K'*B'*P1*A3+K'*B'*P1*B*K+A3'*P2*A3-
A3'*W-W'*A3-P1 ]<0);
F2 =F2 + lmi([A4'*P1*A4+A4'*P1*B*K+K'*B'*P1*A4+K'*B'*P1*B*K+A4'*P2*A4-
A4'*W-W'*A4-P1 ]<0);
F2 =F2 + lmi([A5'*P1*A5+A5'*P1*B*K+K'*B'*P1*A5+K'*B'*P1*B*K+A5'*P2*A5-
A5'*W-W'*A5-P1 ]<0);
F2 =F2 + lmi([A6'*P1*A6+A6'*P1*B*K+K'*B'*P1*A6+K'*B'*P1*B*K+A6'*P2*A6-
A6'*W-W'*A6-P1 ]<0);
F2 =F2 + lmi([A7'*P1*A7+A7'*P1*B*K+K'*B'*P1*A7+K'*B'*P1*B*K+A7'*P2*A7-
A7'*W-W'*A7-P1 ]<0);
F2 =F2 + lmi([A8'*P1*A8+A8'*P1*B*K+K'*B'*P1*A8+K'*B'*P1*B*K+A8'*P2*A8-
A8'*W-W'*A8-P1 ]<0);
F2 =F2 + lmi([A9'*P1*A9+A9'*P1*B*K+K'*B'*P1*A9+K'*B'*P1*B*K+A9'*P2*A9-
A9'*W-W'*A9-P1 ]<0);
F2 =F2 + lmi([A10'*P1*A10+A10'*P1*B*K+K'*B'*P1*A10+K'*B'*P1*B*K+A10'*P2*A10-
A10'*W-W'*A10-P1 ]<0);
```

```
F2 =F2 + lmi([A11’*P1*A11+A11’*P1*B*K+K’*B’*P1*A11+K’*B’*P1*B*K+A11’*P2*A11-
A11’*W-W’*A11-P1 ]<0);
F2 =F2 + lmi([A12’*P1*A12+A12’*P1*B*K+K’*B’*P1*A12+K’*B’*P1*B*K+A12’*P2*A12-
A12’*W-W’*A12-P1 ]<0);
F2 =F2 + lmi([A13’*P1*A13+A13’*P1*B*K+K’*B’*P1*A13+K’*B’*P1*B*K+A13’*P2*A13-
A13’*W-W’*A13-P1 ]<0);
F2 =F2 + lmi([A14’*P1*A14+A14’*P1*B*K+K’*B’*P1*A14+K’*B’*P1*B*K+A14’*P2*A14-
A14’*W-W’*A14-P1 ]<0);
F2 =F2 + lmi([A15’*P1*A15+A15’*P1*B*K+K’*B’*P1*A15+K’*B’*P1*B*K+A15’*P2*A15-
A15’*W-W’*A15-P1 ]<0);
F2 =F2 + lmi([A16’*P1*A16+A16’*P1*B*K+K’*B’*P1*A16+K’*B’*P1*B*K+A16’*P2*A16-
A16’*W-W’*A16-P1 ]<0);
solvesdp(F2)
A0=inv(double(P2))*double(W) ;
P2 = double(P2);
L = sdpvar(2,2);
F3 = lmi([A1’*P1*A1+A1’*P1*B*K+K’*B’*P1*A1+K’*B’*P1*B*K+A1’*P2*A1-
A1’*P2*A0-A0’*P2*A1+A0’*P2*A0-P1 -A1’*P1*B*K-K’*B’*P1*B*K+A1’*P2*A0-
A1’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A1-K’*B’*P1*B*K+A0’*P2*A1-
A0’*P2*A0-C’*L’*P2*A1+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 +lmi([A2’*P1*A2+A2’*P1*B*K+K’*B’*P1*A2+K’*B’*P1*B*K+A2’*P2*A2-
A2’*P2*A0-A0’*P2*A2+A0’*P2*A0-P1 -A2’*P1*B*K-K’*B’*P1*B*K+A2’*P2*A0-
A2’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A2-K’*B’*P1*B*K+A0’*P2*A2-
A0’*P2*A0-C’*L’*P2*A2+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A3’*P1*A3+A3’*P1*B*K+K’*B’*P1*A3+K’*B’*P1*B*K+A3’*P2*A3-
A3’*P2*A0-A0’*P2*A3+A0’*P2*A0-P1 -A3’*P1*B*K-K’*B’*P1*B*K+A3’*P2*A0-
A3’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A3-K’*B’*P1*B*K+A0’*P2*A3-
A0’*P2*A0-C’*L’*P2*A3+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
```

```
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A4’*P1*A4+A4’*P1*B*K+K’*B’*P1*A4+K’*B’*P1*B*K+A4’*P2*A4-
A4’*P2*A0-A0’*P2*A4+A0’*P2*A0-P1 -A4’*P1*B*K-K’*B’*P1*B*K+A4’*P2*A0-
A4’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A4-K’*B’*P1*B*K+A0’*P2*A4-
A0’*P2*A0-C’*L’*P2*A4+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A5’*P1*A5+A5’*P1*B*K+K’*B’*P1*A5+K’*B’*P1*B*K+A5’*P2*A5-
A5’*P2*A0-A0’*P2*A5+A0’*P2*A0-P1 -A5’*P1*B*K-K’*B’*P1*B*K+A5’*P2*A0-
A5’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A5-K’*B’*P1*B*K+A0’*P2*A5-
A0’*P2*A0-C’*L’*P2*A5+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A6’*P1*A6+A6’*P1*B*K+K’*B’*P1*A6+K’*B’*P1*B*K+A6’*P2*A6-
A6’*P2*A0-A0’*P2*A6+A0’*P2*A0-P1 -A6’*P1*B*K-K’*B’*P1*B*K+A6’*P2*A0-
A6’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A6-K’*B’*P1*B*K+A0’*P2*A6-
A0’*P2*A0-C’*L’*P2*A6+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A7’*P1*A7+A7’*P1*B*K+K’*B’*P1*A7+K’*B’*P1*B*K+A7’*P2*A7-
A7’*P2*A0-A0’*P2*A7+A0’*P2*A0-P1 -A7’*P1*B*K-K’*B’*P1*B*K+A7’*P2*A0-
A7’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A7-K’*B’*P1*B*K+A0’*P2*A7-
A0’*P2*A0-C’*L’*P2*A7+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A8’*P1*A8+A8’*P1*B*K+K’*B’*P1*A8+K’*B’*P1*B*K+A8’*P2*A8-
A8’*P2*A0-A0’*P2*A8+A0’*P2*A0-P1 -A8’*P1*B*K-K’*B’*P1*B*K+A8’*P2*A0-
A8’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A8-K’*B’*P1*B*K+A0’*P2*A8-
A0’*P2*A0-C’*L’*P2*A8+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A9’*P1*A9+A9’*P1*B*K+K’*B’*P1*A9+K’*B’*P1*B*K+A9’*P2*A9-
A9’*P2*A0-A0’*P2*A9+A0’*P2*A0-P1 -A9’*P1*B*K-K’*B’*P1*B*K+A9’*P2*A0-
A9’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A9-K’*B’*P1*B*K+A0’*P2*A9-
A0’*P2*A0-C’*L’*P2*A9+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-
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C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 =F3 + lmi([A10’*P1*A10+A10’*P1*B*K+K’*B’*P1*A10+K’*B’*P1*B*K+A10’*P2*A10-A10’*P2*A0-A0’*P2*A10+A0’*P2*A0-P1 -A10’*P1*B*K-K’*B’*P1*B*K+A10’*P2*A0-A10’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A10-K’*B’*P1*B*K+A0’*P2*A10-A0’*P2*A0-C’*L’*P2*A10+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 +lmi([A11’*P1*A11+A11’*P1*B*K+K’*B’*P1*A11+K’*B’*P1*B*K+A11’*P2*A11-A11’*P2*A0-A0’*P2*A11+A0’*P2*A0-P1 -A11’*P1*B*K-K’*B’*P1*B*K+A11’*P2*A0-A11’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A11-K’*B’*P1*B*K+A0’*P2*A11-A0’*P2*A0-C’*L’*P2*A11+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 + lmi([A12’*P1*A12+A12’*P1*B*K+K’*B’*P1*A12+K’*B’*P1*B*K+A12’*P2*A12-A12’*P2*A0-A0’*P2*A12+A0’*P2*A0-P1 -A12’*P1*B*K-K’*B’*P1*B*K+A12’*P2*A0-A12’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A12-K’*B’*P1*B*K+A0’*P2*A12-A0’*P2*A0-C’*L’*P2*A12+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 + lmi([A13’*P1*A13+A13’*P1*B*K+K’*B’*P1*A13+K’*B’*P1*B*K+A13’*P2*A13-A13’*P2*A0-A0’*P2*A13+A0’*P2*A0-P1 -A13’*P1*B*K-K’*B’*P1*B*K+A13’*P2*A0-A13’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A13-K’*B’*P1*B*K+A0’*P2*A13-A0’*P2*A0-C’*L’*P2*A13+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 + lmi([A14’*P1*A14+A14’*P1*B*K+K’*B’*P1*A14+K’*B’*P1*B*K+A14’*P2*A14-A14’*P2*A0-A0’*P2*A14+A0’*P2*A0-P1 -A14’*P1*B*K-K’*B’*P1*B*K+A14’*P2*A0-A14’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A14-K’*B’*P1*B*K+A0’*P2*A14-A0’*P2*A0-C’*L’*P2*A14+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-C’*L’*P2*A0-P2 C’*L’*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 + lmi([A15’*P1*A15+A15’*P1*B*K+K’*B’*P1*A15+K’*B’*P1*B*K+A15’*P2*A15-A15’*P2*A0-A0’*P2*A15+A0’*P2*A0-P1 -A15’*P1*B*K-K’*B’*P1*B*K+A15’*P2*A0-A15’*P2*L*C-A0’*P2*A0+A0’*P2*L*C zeros(2,2); -K’*B’*P1*A15-K’*B’*P1*B*K+A0’*P2*A15-A0’*P2*A0-C’*L’*P2*A15+C’*L’*P2*A0 K’*B’*P1*B*K+A0’*P2*A0-A0’*P2*L*C-

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C'*L'*P2*A0-P2 C'*L'*P2;zeros(2,2) P2*L*C -P2] <0) ;
F3 = F3 + lmi([A16'*P1*A16+A16'*P1*B*K+K'*B'*P1*A16+K'*B'*P1*B*K+A16'*P2*A16-
A16'*P2*A0-A0'*P2*A16+A0'*P2*A0-P1 -A16'*P1*B*K-K'*B'*P1*B*K+A16'*P2*A0-
A16'*P2*L*C-A0'*P2*A0+A0'*P2*L*C zeros(2,2); -K'*B'*P1*A16-K'*B'*P1*B*K+A0'*P2*A16-
A0'*P2*A0-C'*L'*P2*A16+C'*L'*P2*A0 K'*B'*P1*B*K+A0'*P2*A0-A0'*P2*L*C-
C'*L'*P2*A0-P2 C'*L'*P2;zeros(2,2) P2*L*C -P2] <0) ;
solvesdp(F3)
L=double(L)
```

Vita

Name: Mohammed Abdulrahman Alsuwaii

Date of Birth: July 1976

Nationality: Saudi

Email: mhd62976@gmail.com

Education:

Bachelor Degree of Electrical Engineering from King Fahd University of Petroleum & Minerals (June 2000) , Dhahran, Saudi Arabia.

Experience:

Aug 2000 - July 2002: Worked in AMCDE(Azmi & Abdulla Consulting Engineering Company) as design engineer .

Since Aug 2002 : worked in Saudi Electricity Company (SEC) as electrical engineer in Dammam